UNIVERSITY OF BAYREUTH

Structure of the Expected Exchange Rates within the framework of the Model of Cox, Ingersoll, and Ross for Modelling the Term Structure

Thesis of Sebastian Horlemann

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I hereby declare that I worked on this thesis on my own and only used the indicated aid and literature.

Bayreuth, 18.02.2005

Sebastian Horlemann

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The following table lists the most important symbols used throughout this thesis:

Symbol	Interpretation
t	current time; set to 0 (arbitrary)
s	future time; $s > t$
T	future time; date of maturity of zero bond
$T - t$	time-to-maturity of zero bond at current time t
κ	speed of adjustment of spot rate
θ	long-term value of spot rate
σ^2	interest rate variance
λ	market risk value
r, $r(t)$	current interest rate
$r(s)$	interest rate at future time s
$(\cdot)^*$	the respective variable of country Y
$P(r, t, T)$	price of zero bond at time t with time-to-maturity $T - t$ and spot rate r
$R(r, t, T)$	yield of zero bond at time t with time-to-maturity $T - t$ and spot rate r
A, B	determine the price and yield of a zero bond respectively
ϕ_1, ϕ_2, ϕ_3	determine the price and yield of a zero bond respectively
E_t	current exchange rate; set to 1 (arbitrary)
E_s	exchange rate at future time s
E_s^e	expected exchange rate at future time s
Δ	length of period; set to 0.01 in the latter part of this thesis (arbitrary)

Chapter 1

Introduction

1.1 Motivation

Exchange rates play a very important role in many areas. Economic situations, investment decisions and even the cost of travelling are influenced by the cost of foreign currency. Beside the spot exchange rate the expectation of its future value has important implications on many decisions. The pressure of an appreciation and depreciation, respectively, on the exchange rate influences the movement of capital and investment. Investments and their yields in the home or foreign country depend critically on the expected exchange rate in the future. The exchange rate risk determines the risk premium for investments in a foreign country and has led to many financial instruments used to hedge against this risk. Furthermore, the costs and, as a result, the competitiveness of exported and imported goods are fundamentally dependent on the exchange rate. In addition to that, the exchange rate politics of a central bank are influenced by the expectations of the future values of the exchange rate. These examples and further considerations emphasize the importance of the exchange rates and the expectation of their future development. Considering all these facts it is not surprising that researchers spend a great effort to explain the mechanism of determination of the spot exchange rate, the future expectation and their characteristics.

In this thesis, I present possibilities of evolving paths of expected exchange rates and investigate their particular structure and behavior. The model of Cox, Ingersoll, and Ross [6] serves as a cornerstone for modelling the term structure and its implications. As interest rates interact with exchange rates and vice versa, an approach of investigating the expectations needs to take the term structure into consideration. I hold the view that the model of Cox, Ingersoll, and Ross [6] is perfectly suited for this task. Contrary to other models, the expectation of the exchange rate is not only investigated at one particular future time, but the whole path of expected values is examined. This approach takes into consideration that, similar to the term structure, there exist investors with different preferences regarding the length and other characteristics of investment opportunities. Consequently, various expectations at different future times are formed.

After introducing certain mathematical definitions and theorems used throughout the whole thesis and necessary for a deeper understanding of the papers of Cox, Ingersoll, and Ross [5], [6], the theory of the term structure is presented. First of all, the general equilibrium model is introduced. After that, this model is used to evolve the term structure. In Section 3.3, the characteristics are analyzed and visualized. In Chapter 4, expectations of future values of the exchanges rates are evaluated. It is important to take into consideration that there are several possibilities of forming expectations. In Chapter 5, another approach of calculating the expected exchange rates and their depreciation rates respectively is presented. Chapter 6 deals with the question of how the expectations and changes of the path of expectations due to changes in the fundamental factors influence the determination of the spot exchange rate. Contrary to more simple models determining the current exchange rate, e.g. based on the interest rate parity, the influence of the whole path of expectations on the spot exchange rate is investigated and may, therefore, represent an extension and a more realistic approach. Finally, various examples are presented, analyzed and visualized in Chapter 7.

The CD-ROM includes all MATLAB source files mentioned throughout this thesis as well as the thesis itself.

At this point, I would like to thank Dr. Christian Bauer for the support and the co-operation. Furthermore, I would like to thank Professor Dr. Lars Gruene and Professor Dr. Bernhard Herz. This thesis is dedicated to my parents.

1.2 Notions and Translations

In my thesis, I use several notions for economic variables. In this section, I introduce the nomenclature and the meaning of the various terms.

First of all, the notion i is used to describe the *interest rate*. The *interest* is the payment made by a borrower for the use of money. It is usually calculated as a percentage of the capital borrowed. This percentage is called the *interest rate*. One can distinguish between *discrete* payments of interest, e.g. every year, and *continuous* payments.

The *yield-to-maturity* is the annualized rate of return in percentage terms on a fixed income instrument such as a bond.
A *discount bond* is a bond, which is sold at a price below its *face value* and returns its face value at *maturity*.

If one considers continuous payments of interest and compound interest, the connection between the price of a bond and its yield-to-maturity can be expressed as follows:

$$FV \quad = \quad SP \cdot e^{R\cdot(T-t)},$$

where, FV is the future value and SP the *starting principal* of the bond. Furthermore, R is the yield-to-maturity and $T-t$ the *time-to-maturity* if T is the date of maturity and t the current time. The expression

$$e^{R\cdot(T-t)}-1,$$

is called the *total return*. It is given in percentage terms. If a bond's yield-to-maturity equals R and the time-to-maturity is $T-t$, the total return is expressed by R^{TR}. Consequently,

$$R^{TR} \quad = \quad e^{R\cdot(T-t)}-1.$$

The *spot rate* at time s is the rate that applies to an infinitesimally short period of time at time s. The notions spot rate and *current interest rate* are synonymous.

The notion *long-term yield* indicates the yield for a bond with a relatively long time-to-maturity and *short-term yield* indicates the yield for a bond with a relatively short time-to-maturity. The yield *in the long-run* indicates the yield for a bond purchased at a relatively distant future point of time, where the yield *in the short-run* indicates the yield for a bond purchased in the relatively near future.

The *exchange rate* is the price of one unit of the foreign currency valued in the domestic currency. This is called the *price quotation*. The *spot exchange rate* is the exchange rate at a particular point of time.

Furthermore, I have included a table with translations from English into German, so that readers may understand the use of the terms more easily.

English	German
discount bond	Nullkuponanleihe
in the long-run	auf lange Sicht
in the short-run	auf kurze Sicht
interest	Zinsen
interest rate	Zins (in Prozent)
long-term	langfristig
short-term	kurzfristig
spot rate	aktueller Zins (in Prozent)
maturity	Faelligkeit
time-to-maturity	Laufzeit
yield-to-maturity	jaehrliche Rendite (in Prozent)
price quotation	Preisnotierung
total return	Vermoegenszuwachs ueber die gesamte Laufzeit (in Prozent)
starting principal	Kapitalsumme zu Beginn
face value	Nominalwert

Chapter 2

Mathematical Preliminaries

2.1 Probability Spaces, Random Variables and Stochastic Processes

To understand the calculations within this thesis, we need to find reasonable mathematical notions corresponding to the quantities mentioned and mathematical models for the problems. To this end, a mathematical model for a random quantity has to be introduced. Before defining what a *random variable* is, it is helpful to recall some concepts from general probability theory. The reader is referred to e.g. Bol [1] or Øksendal [13].

Definition 2.1 *If* Ω *is a given set, then a* σ -algebra $\mathcal{F}$ on Ω *is a family* $\mathcal{F}$ *of subsets of* Ω *with the following properties:*

(i) $\emptyset \in \mathcal{F}$

(ii) $F \in \mathcal{F} \Rightarrow \mathcal{F}^C \in \mathcal{F}$, *where* $F^C = \Omega \setminus F$ *is the complement of* F *in* Ω

(iii) $A_1, A_2, \ldots \in \mathcal{F} \Rightarrow A := \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$

The pair $(\Omega, \mathcal{F})$ *is called a* measurable space*. A* probability measure P *on a measurable space* $(\Omega, \mathcal{F})$ *is a function* $P : \mathcal{F} \to [0, 1]$*, such that:*

(a) $P(\emptyset) = 0, P(\Omega) = 1$

(b) *if* $A_1, A_2, \ldots \in \mathcal{F}$ *and* $\{A_i\}_{i=1}^{\infty}$ *is* disjoint*, then*

$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$

The triple $(\Omega, \mathcal{F}, P)$ *is called a* probability space*. It is called a* complete probability space *if* $\mathcal{F}$ *contains all subsets* G *of* Ω *with* P*-outer measure zero, i.e. with*

$$P^*(G) := \inf\{P(F); F \in \mathcal{F}, G \subset F\} = 0.$$

It is obvious that any probability space can be made complete simply by adding to $\mathcal{F}$ all sets of outer measure zero and by extending P accordingly. In the following we let $(\Omega, \mathcal{F}, P)$ denote a given complete probability space.

A well known example of a σ-algebra is the *Borel* σ-algebra containing all open sets, all closed sets, all countable unions of closed sets, all countable intersections of such countable unions, etc.

Definition 2.2 *If* $(\Omega, \mathcal{F}, P)$ *is a given probability space, then a function* $Y : \Omega \to \mathbb{R}^n$ *is called* $\mathcal{F}$-measurable *if*

$$Y^{-1}(U) := \{\omega \in \Omega; Y(\omega) \in U\} \in \mathcal{F}$$

for all open sets $U \in \mathbb{R}^n$*.*

Now it is possible to define a *random variable*:

Definition 2.3 *A* random variable X *is an $\mathcal{F}$-measurable function $X : \Omega \rightarrow \mathbb{R}^n$.*

Every random variable induces a *probability measure* $\mu_X(B) = P(X^{-1}(B))$. μ_X is called the *distribution* of X.

Moreover, the *mean* of a random variable and the *independency* of various random variables are defined the following way:

Definition 2.4 *If $\int_\Omega \mid X(\omega) \mid dP(\omega) < \infty$, then the number*

$$E(X) := \int_\Omega X(\omega)dP(\omega) = \int_{\mathbb{R}^n} x d\mu_X(x) \tag{2.1}$$

is called the expectation *of X.*

More generally, if $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is Borel measurable and $\int_\Omega |f(X(\omega))|dP(\omega) < \infty$, then we have

Definition 2.5

$$E(f(X)) := \int_\Omega f(X(\omega))dP(\omega) = \int_{\mathbb{R}^n} f(x)d\mu_X(x). \tag{2.2}$$

Definition 2.6 *A collection of random variables $\{X_i : i \in I\}$ is* independent *if the collection of generated σ-algebras $\mathcal{H}_{X_i}$ is* independent*, that is $P(H_{i_1} \cap \ldots \cap H_{i_k}) = P(H_{i_1}) * \ldots * P(H_{i_k})$ for all choices of $H_{i_1} \in \mathcal{H}_{X_{i_1}}, \ldots, H_{X_{i_k}} \in \mathcal{H}_{i_k}$ with different indices $i_1, \ldots, i_k$. The σ-algebra* generated *by $\mathcal{U}$, $\mathcal{H}_\mathcal{U}$, is the smallest σ-algebra containing $\mathcal{U}$.*

Furthermore, we have to deal with *stochastic processes*. The definition is as follows:

Definition 2.7 *A* stochastic process *is a parameterized collection of random variables*

$$\{X_t\}_{t \in T}$$

defined on a probability space $(\Omega, \mathcal{F}, P)$ *and assuming values in* $\mathbb{R}^n$.

The parameter space T is usually the half-line $[0, \infty)$. Note that for each $t \in T$ fixed we have a random variable

$$\omega \to X_t(\omega); \quad \omega \in \Omega.$$

On the other hand, fixing $\omega \in \Omega$ we can consider the function

$$t \to X_t(\omega); \quad t \in T,$$

which is called a *path* of X_t.

It may be useful for the intuition to think of t as 'time' and each ω as an individual 'particle' or 'experiment'. With this picture $X_t(\omega)$ would represent the position (or result) at time t of the particle (or experiment) ω.

For later purposes we need the *Brownian motion*, B_t. It is a particular stochastic process. For further information about the Brownian motion and its properties the reader is referred to Yeh [15].

2.2 An Introduction to Stochastic Differential Equations (SDE)

2.2.1 General Form of a Stochastic Differential Equation

If we allow for some randomness in some of the coefficients of a differential equation, we often obtain a more realistic mathematical model of a particular situation. For example, a *stochastic differential equation* may take the form:

$$\frac{dX_t}{dt} = b(t, X_t) + \sigma(t, X_t)W_t, \tag{2.3}$$

where b and σ are some given functions. As one can see, we allow for some randomness by introducing a 'noise' term represented by the stochastic process W_t.

We expect the stochastic process W_t to fulfill the following characteristics:

(i) $t_1 \neq t_2 \Rightarrow W_{t_1}$ and W_{t_2} are independent

(ii) $\{W_t\}$ is stationary, i.e. the (joint) distribution of $\{W_{t_1+t} \ldots W_{t_k+t}\}$ does not depend on t

(iii) $E(W_t) = 0$ for all t

If we rewrite (2.3) by replacing the W_t-notation by a suitable stochastic process $\{V_t\}_{t\geq 0}$ and if we take into account the assumptions (i)-(iii) on the stochastic process, V_t can be identified by the Brownian motion B_t, the stochastic differential equation can be written as:

$$X_k = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s. \tag{2.4}$$

The existence, in a certain sense, of the latter integral of (2.4) will be proven in the remainder of this chapter.

2.2.2 Itô's Integral and Itô's Formula

In this part, we will introduce the *Itô integral* and the *Itô formula* for problems dealing with *1-dimensional stochastic differential equations*. However, straightforward calculations lead to an extension of the definitions for n-dimensional cases. The reader is referred to Øksendal [13].

It can be shown that for a set of *elementary functions*, which take the form

$$\phi(t, \omega) = \sum_j e_j(\omega) \cdot \chi_{[t_j, t_{j+1})}(t),$$

where χ denotes the characteristic (indicator) function, the integral $\int_S^T \phi(t, \omega)dB_t(\omega)$ can be defined as follows:

$$\int_S^T \phi(t, \omega)dB_t(\omega) = \sum_{j\geq 0} e_j(\omega) \cdot [B_{t_{j+1}} - B_{t_j}](\omega),$$

where

$$t_k = t_k^n = \left\{ \begin{array}{ccl} k \cdot 2^{-n} & if & S \leq k \cdot 2^{-n} \leq T \\ S & if & k \cdot 2^{-n} < S \\ T & if & k \cdot 2^{-n} > T \end{array} \right\}.$$

Note that - unlike the Riemann-Stieltjes integral - it does make a difference here what points t_j we choose. The choice of using the left end point of the interval leads to the Itô integral.

Furthermore, there exists a class $\mathcal{V}(S,T)$ of functions (for further information see Øksendal [13]) for which the Itô integral can be defined. It can be shown that for any function f of this class there exists a sequence $\{\phi_n\}$ of elementary functions, such that:

$$E\Big[\int_S^T (f-\phi_n)^2 dt\Big] \to 0 \quad as \quad n \to \infty.$$

As a result, the Itô integral can be defined:

Definition 2.8 *Let* $f \in \mathcal{V}(S,T)$. *Then the* Itô integral *of* f *(from* S *to* T*) is defined by:*

$$\int_S^T f(t,\omega) dB_t(\omega) = \lim_{n\to\infty} \int_S^T \phi_n(t,\omega) dB_t(\omega),$$

where $\{\phi_n\}$ *is a sequence of elementary functions, such that*

$$E\Big[\int_S^T (f(t,\omega) - \phi_n(t,\omega))^2 dt\Big] \to 0 \quad as \quad n \to \infty.$$

In the latter part, we may refer to the following property of the Itô integral:

$$E\Big[\int_S^T f dB_t\Big] = 0 \quad for \quad f \in \mathcal{V}(0,T). \tag{2.5}$$

If X_t is an *Itô process* (see Øksendal [13]), than

$$X_t = X_0 + \int_0^t u(s,\omega) ds + \int_0^t v(s,\omega) dB_s$$

can be written in the shorter differential form

$$dX_t = udt + vdB_t.$$

Now we will introduce the Itô formula which is easier to handle than the Itô integral for calculations.

Definition 2.9 *Let X_t be an Itô process given by:*

$$dX_t = udt + vdB_t.$$

Let $g(t,x) \in C^2([0,\infty) \times R)$. Then $Y_t = g(t, X_t)$ is again an Itô process, and

$$dY_t = \frac{\partial g}{\partial t}(t, X_t)dt + \frac{\partial g}{\partial x}(t, X_t)dX_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t, X_t)(dX_t)^2,$$

where $(dX_t)^2 = (dX_t)(dX_t)$ is computed according to the rules

$$dt \cdot dt = dt \cdot dB_t = dB_t \cdot dt = 0, \quad dB_t \cdot dB_t = dt.$$

As mentioned above, the definitions can be extended to define the *multidimensional Itô integral* and the *multidimensional Itô formula*.

For an example of usage for solving a stochastic differential equation see Appendix A.

2.2.3 Stochastic Control Problems and the Hamilton-Jacobi-Bellman Equation

In this context with control problems which allow for some randomness, the *Hamilton-Jacobi-Bellman equation* offers a solution method. However, before stating the theorem a few properties of the so called *Itô diffusion* need to be investigated.

Definition 2.10 *A time homogeneous* Itô diffusion *is a stochastic process*

$X_t(\omega) = X(t,\omega) : [0,\infty) \times \Omega \to \mathbb{R}^n$ *satisfying a stochastic differential equation of the form*

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t, \quad t \geq s; \quad X_s = x,$$

where B_t is m-dimensional Brownian motion and b, σ satisfy particular conditions necessary for the existence and uniqueness of a solution of a stochastic differential equation.[1]

Usually, b is called the *drift coefficient* and σ, the *diffusion coefficient.*

For further investigations, we need to describe the observed behavior of a stochastic process. For this purpose we define:

Definition 2.11 *Let $B_t(\omega)$ be n-dimensional Brownian motion. Then we define*

$\mathcal{F}_t = \mathcal{F}_t^{(n)}$ *to be the σ-algebra generated by the random variables $\{B_i(s)\}_{1\leq i\leq n, 0\leq s\leq t}$.*

In other words, $\mathcal{F}_t$ is the smallest σ-algebra containing all sets of the form

$$\{\omega; B_{t_1}(\omega) \in F_1, \ldots, B_{t_k}(\omega) \in F_k\},$$

where $t_j \leq t$ and $F_j \subset \mathbb{R}^n$ are Borel sets, $j \leq k = 1, 2, \ldots$

One often thinks of $\mathcal{F}_t$ as the history of B_s up to time t. Intuitively, a function h is $\mathcal{F}_t$-measurable if the value of $h(\omega)$ can be decided from the values of $B_s(\omega)$ for $s \leq t$. Note that $\mathcal{F}_s \subset \mathcal{F}_t$ for $s < t$, that is, $\{\mathcal{F}_t\}$ is increasing.

Straightforward calculations prove that an Itô diffusion satisfies the important *Markov property*, stating that the future behavior of the process given what has

[1]For further information the reader is referred to Øksendal [13].

happened up to time t is the same as the behavior obtained when starting the process at X_t. For further information the reader is referred to Øksendal [13].

Moreover, the *strong Markov property* states that the *Markov property* holds if the time t is replaced by a random time $\tau(\omega)$ of a more general type called *stopping time*:

Definition 2.12 *Let $\{\mathcal{N}_t\}$ be an increasing family of σ-algebras. A function*

$\tau : \Omega \to [0, \infty]$ *is called a* (strict) stopping time *with respect to* $\{\mathcal{N}_t\}$ *if*

$$\{\omega; \tau(\omega) \leq t\} \in \mathcal{N}_t$$

for all $t \geq 0$.

For an open $U \subset \mathbb{R}^n$ the *first exit time*

$$\tau_U := inf\{t > 0; X_t \in \mathbb{R}^n \setminus U\}$$

is a stopping time with respect to $\mathcal{M}_t$, which is the σ-algebra generated by $\{X_r; r \leq t\}$.

For solving stochastic control problems it is fundamental to associate a second order partial differential operator A to an Itô diffusion. The basic connection between A and X_t is the generator of the process X_t. One can show that for an Itô diffusion and a function $f \in C_0^2(\mathbb{R}^n)$ the generator A satisfies

$$Af(x) = \sum_i b_i(x) \frac{\partial f}{\partial x_i} + \frac{1}{2} \sum_{i,j} (\sigma \sigma^T)_{i,j}(x) \frac{\partial^2 f}{\partial x_i \partial x_j}. \tag{2.6}$$

Suppose that the state of a system at time t is described by an Itô process X_t of the form:

$$dX_t = dX_t^u = b(t, X_t, u_t)dt + \sigma(t, X_t, u_t)dB_t, \tag{2.7}$$

where $X_t \in \mathbb{R}^n, b : \mathbb{R} \times \mathbb{R}^n \times U \to \mathbb{R}^n, \sigma : \mathbb{R} \times \mathbb{R}^n \times U \to \mathbb{R}^{n \times m}$ and B_t is m-dimensional Brownian motion. Here $u_t \in U \subset \mathbb{R}^k$ is a parameter whose value we can choose in the given Borel set U at any instant t in order to control the process X_t. Thus $u_t = u(t, \omega)$ is a stochastic process. Since our decision at time t must be based upon what has happened up to time t, the function $\omega \to u(t, \omega)$ must at least be measurable with respect to $\mathcal{F}_t^{(m)}$.

Let $f : \mathbb{R} \times \mathbb{R}^n \times U \to \mathbb{R}$ and $g : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be given continuous functions, let G be a fixed domain in $\mathbb{R} \times \mathbb{R}^n$ and let $\hat{T}$ be the first exit time after s from G for the process $\{X_r^{s,x}\}_{r \geq s}$, where $X_r^{s,x}$ is the solution of (2.7) at time r with an initial value $x = X_s^{s,x}$ at time s. Substituting $Y_t = (s+t, X_{s+t}^{s,x})$, where $y = (s,x)$ represents the starting point of the process Y_t, one can introduce a performance function

$$J^u(y) = E^y\Big[\int_0^{\tau_G} f^{u_t}(Y_t)dt + g(Y_{\tau_G})\chi_{\{\tau_G < \infty\}}\Big], \tag{2.8}$$

where $\tau_G = \hat{T} - s$. The performance function represents the value of the function to be maximized dependent on the value of (s,x), that is the initial value x at time s, and the choice of the control. The problem is - for each $y \in G$ - to find the number $\Phi(y)$ and a control $u^* = u^*(t,\omega) = u^*(y,t,\omega) \in \mathcal{A}$, such that:

$$\Phi(y) := \sup_{u(t,\omega)} J^u(y) = J^{u^*}(y), \tag{2.9}$$

where the supremum is taken over a given family $\mathcal{A}$ of *admissible* controls. Such a control u^* - if it exists - is called an *optimal control* and Φ is called the *optimal performance*.

Functions $u(t,\omega)$ of the form $u(t,\omega) = u_0(t, X_t(\omega))$ for some function $u_0 : \mathbb{R}^{n+1} \to U \subset \mathbb{R}^k$ are called *Markov controls*. We only consider *Markov controls* $u = u(t, X_t(\omega))$.

For $v \in U$ and $\phi \in C_0^2(\mathbb{R} \times \mathbb{R}^n)$ we define

$$(L^v\phi)(y) = \frac{\partial \phi}{\partial s}(y) + \sum_i b_i(y,v)\frac{\partial \phi}{\partial x_i} + \frac{1}{2}\sum_{i,j}(\sigma\sigma^T)_{i,j}(y,v)\frac{\partial^2 \phi}{\partial x_i \partial x_j} \quad , \tag{2.10}$$

where $y = (s,x)$ and $x = (x_1, \cdots, x_n)$. Then for each choice of the function u the solution $Y_t = Y_t^u$ is an Itô diffusion with generator A (see (2.6)) given by

$$(A\phi)(y) = (L^{u(y)}\phi)(y).$$

It can be shown that reducing the investigation on Markov controls is not too restrictive.

Now one can state the *Hamilton-Jacobi-Bellman Equation*:

Theorem 2.13 (The Hamilton-Jacobi-Bellman Equation)

Define

$$\Phi(y) = sup\{J^u(y); u = u(Y) \quad Markov \quad control\}.$$

Suppose that an optimal Markov control u^* exists. Under certain conditions[2] there is

$$\sup_{\upsilon \in U} \{ f^{\upsilon}(y) + (L^{\upsilon}\Phi(y)) \} = 0 \tag{2.11}$$

for all $y \in G$ and

$$\Phi(y) = g(y)$$

for all $y \in \partial G$.

The supremum in (2.11) is obtained if $\upsilon = u^*(y)$ where $u^*(y)$ is optimal. In other words,

$$f(y, u^*(y)) + (L^{u^*(y)}\Phi)(y) = 0$$

for all $y \in G$.

It can be shown that the condition above is also *sufficient*.

[2] For further information the reader is referred to Øksendal [13].

Chapter 3

The Term Structure of an Economy

In this chapter, a theory of the term structure of interest rates based on a general equilibrium model of asset prices is introduced. Both models are treated in the papers of Cox, Ingersoll, and Ross [5], [6]. It is important to point out that in Section 3.1 and 3.2 only an outline of the results and ideas is presented. Furthermore, I emphasize that the major interest is paid on those results which are necessary for the problems dealt with within this thesis. Hence, the results of Section 3.2, especially those dealing with the current interest rate and the term structure, which are the cornerstones of the research done in Chapter 4, 5, and 6, are more important than those of Section 3.1. To gain deeper understanding of the results and proofs of these two papers, I hold the view that the interested reader needs to study the rather complicated models more intensively. I confine myself to those results which serve as the background of my investigations.

3.1 An Intertemporal General Equilibrium Model of Asset Prices (CIRI)

In this paper by Cox, Ingersoll and Ross [5], a general equilibrium asset pricing model for use in applied research is developed. The main result is the endogenously determined price of any asset in terms of the underlying variables in the economy. It follows from the solution of a partial differential equation, which needs to be satisfied by the asset prices. Several assumptions are made to describe a simple, however easily extended, model of the economy.

First of all, it is assumed that there is a only a single physical good, which may be either allocated to consumption or investment. Furthermore, all existing values are expressed in terms of units of this good.

The production possibilities are described by a system of stochastic differential equations transforming an investment of a vector η of amounts of the good in the n linear production processes. The processes take the form:

$$d\eta(t) = I_\eta \alpha(Y,t)dt + I_\eta G(Y,t)d\omega(t). \tag{3.1}$$

The vector of expected rate of return on η is described by $\alpha(Y,t)$.[1]

In the equation above and in the rest of this chapter $\omega(t)$ is a $(n+k)$ dimensional Wiener process or Brownian motion.

Additionally, the vector of the state variables describing the status of the economy is also determined by a system of stochastic differential equations. It takes the following form:

$$dY(t) = \mu(Y,T)dt + S(Y,t)d\omega(t). \tag{3.2}$$

The k-dimensional vector of expected rate of return on $Y(t)$ is described by $\mu(Y,t)$.[2] Both Y and the joint process (η, Y) are *Markov*.

Consequently, this framework includes both uncertain production and random technological change. The latter leads to a change in the status of the economy. As a result, the probability distribution of the current output depends on the

[1] $\alpha(Y,t)$ is a bounded n-dimensional vector valued function of Y and t. I_η is an $n \times n$ diagonal matrix valued function of η whose ith diagonal element is the ith component of η. $G(Y,t)$ is a bounded $n \times (n+k)$ matrix valued function of Y and t.

[2] $\mu(Y,t)$ is k-dimensional.

current level of the state variables Y, which are themselves changing randomly over time and thus will determine the future production opportunities.

The economy's individuals have different opportunities to invest their wealth. Besides investing in physical production through firms, there is a market for instantaneous borrowing and lending at an interest rate r, which is the riskless alternative to investing in contingent claims to amounts of the good. These securities are specified by all payoffs which may be received from that claim. These payoffs may depend on the state variables and on the level of aggregate wealth. Therefore, one can write the stochastic differential equation governing the movement of the value of claim i, F^i, as

$$dF^i = (F^i\beta_i - \delta_i)dt + F^i h_i d\omega(t). \tag{3.3}$$

To express the dependency of the payout on wealth and the status variables, one can write δ_i as $\delta(W,Y)_i$. The total mean return on claim i, $\beta_i F^i$, is defined as the payout received, δ_i, plus the mean price change, $F^i\beta_i - \delta_i$. That is, expecting a payoff of δ_i, the total mean return is reduced by that amount.

A fixed number of homogenous individuals seeks to maximize an objective function of the form:

$$E\int_t^{t'} U[C(s), Y(s), s]ds. \tag{3.4}$$

U is a *von Neumann-Morgenstern* utility function dependent on the consumption flow $C(s)$ and the state of the economy $Y(s)$. In addition to that, adjustment or transactions costs are neglected.

Cox, Ingersoll, and Ross assume $(n + k)$ basis opportunities for investment in physical production or contingent claims. Additionally, the $(n + k + 1)$st is the opportunity in riskless lending or borrowing. It is pointed out that a greater number of opportunities can be easily obtained by a linear combination of the basis. If W represents the current amount of wealth, the budget constraint of the maximizing problem can be written as

$$\begin{aligned} dW &= \Big[\sum_{i=1}^{n} a_i W(\alpha_i - r) + \sum_{i=1}^{k} b_i W(\beta_i - r) + rW - C\Big]dt \\ &+ \sum_{i=1}^{n} a_i W\Big(\sum_{j=1}^{n+k} g_{ij} d\omega_j\Big) + \sum_{i=1}^{k} b_i W\Big(\sum_{j=1}^{n+k} h_{ij} d\omega_j\Big) \\ &\equiv W\mu(W)dt + W\sum_{j=1}^{n+k} q_j d\omega_j, \end{aligned} \tag{3.5}$$

where $a_i W$ is the amount of wealth invested in the ith process, and $b_i W$ is the amount of wealth invested in the ith contingent claim.

Taking into account that a_i represents investment in physical processes, the value of a_i needs to be nonnegative. Moreover, negative consumption does not make sense. Further assumptions allow for the application of standard results from stochastic control theory. As a result, the stochastic control problem of maximizing (3.4), where the system is described by (3.2) and (3.5), can be easily solved using the *Hamilton-Jacobi-Bellmann equation* leading to a system of necessary and sufficient conditions for the maximization of

$$\Psi \equiv L^{\nu} J + U \tag{3.6}$$

as a function of C, a, b, where J is an indirect utility function and $\nu \in V$. V is a class of admissable feedback controls.[3]

Considering the methods of *Kuhn-Tucker* these conditions can be written as:

$$\begin{aligned}
\Psi_C &= U_C - J_W \leq 0 & (3.7)\\
C\Psi_C &= 0 & (3.8)\\
\Psi_a &= [\alpha - r1]WJ_W + [GG^T a + GH^T b]W^2 J_{WW} & \\
&+ GS^T W J_{WY} \leq 0 & (3.9)\\
a^T \Psi_a &= 0 & (3.10)\\
\Psi_b &= [\beta - r1]WJ_W + [HG^T a + HH^T b]W^2 J_{WW} & \\
&+ HS^T W J_{WY} = 0, & (3.11)
\end{aligned}$$

where 1 is a $(k \times 1)$ unit vector.

W and J are the current wealth and indirect utility function respectively. The optimal values of $\hat{C}$, $\hat{a}$,and $\hat{b}$ are functions of only W, Y, and t.

The economy's equilibrium is reached when the current wealth is only invested in physical production, that is, $\sum a_i = 1$ and $b_i = 0$ for all i. This straightforward definition follows from the assumption of homogenous individuals. Furthermore, the individual chooses $\hat{C}$, $\hat{a}$, and $\hat{b}$ taking r, α,and β as given. Using the equations (3.9)-(3.10) one can show that with $b = 0$ the values of a and r and, consequently, β can be determined. The value of α is exogenously given. The equilibrium interest rate r, the equilibrium expected returns on the contingent claim β, the total production plan a, and the total consumption plan C are determined endogenously. The equilibrium values are calculated by investigating two related problems. First of all, it is assumed that there is only investment in physical production. The equilibrium value for a^* and Lagrangian multiplier λ^* can be used

[3] For further information about the application of the stochastic control theory in this particular case see Appendix B.1.

to evolve the value for r^* for the extended problem with borrowing and lending. Note that both problems meet the requirement of an equilibrium, that is, $b_i = 0$ for all i. The equilibrium interest rate can be written as:

$$r(W,Y,t) = \frac{\lambda^*}{WJ_W} \tag{3.12}$$

$$= a^{*T}\alpha - \Big(\frac{-J_{WW}}{J_W}\Big)\Big(\frac{var(W)}{W}\Big) - \sum_{i=1}^{k}\Big(\frac{-J_{WY_i}}{J_W}\Big)\Big(\frac{cov(W,Y_i)}{W}\Big), \tag{3.13}$$

where $cov(W, Y_i)$ stands for the covariance of changes in optimally invested wealth with changes in the state variable Y_i and similarly for $var(W)$. The expected rate of return on optimally invested wealth is $a^{*T}\alpha$. As one can see, the equilibrium interest rate may be either less or greater than $a^{*T}\alpha$, even though all individuals are risk averse to gambles on consumption paths. Although investment in the production processes exposes an individual to uncertainty about the output received, it may also allow the investor to hedge against the risk of less favorable changes in technology. An individual investing only in locally riskless lending would be unprotected against this latter risk. In general, either effect may dominate. A more intuitive interpretation of the equilibrium interest rate states that r equals the negative expected rate of change in the marginal utility of wealth.

The authors now turn to the determination of the equilibrium expected return on any contingent claim. One can show that by using the system (3.7)-(3.11) and the Itô's formula the ith value is given by:

$$(\beta_i - r)F^i = [\phi_W \quad \phi_{Y_1} \quad \dots \quad \phi_{Y_k}][F^i_W \quad F^i_{Y_1} \quad \dots \quad F^i_{Y_k}]^T. \tag{3.14}$$

The equilibrium expected return for any contingent claim can thus be written as the riskfree return plus a linear combination of the first partials of the asset price with respect to W and Y. The coefficients ϕ_W and ϕ_{Y_i}, respectively, can be interpreted as factor risk premiums. The factor risk premium of the jth factor is defined as the excess expected rate of return on a security or portfolio which has only the risk of the jth factor.

It can also be shown that

$$\beta_i - r = \frac{-cov(F^i, J_W)}{F^i J_W}. \tag{3.15}$$

That is, the excess expected rate of return on the ith contingent claim is equal to the negative of the covariance of its rate or return with the rate of change in the marginal utility of wealth. Just as one would expect, individuals are willing to accept a lower expected rate of return on securities which tend to pay off more highly when marginal utility is greater. Hence, in equilibrium such securities will have a lower total risk premium. Now it is possible to derive the fundamental valuation equation for the contingent claims. Considering the results from above, the price of any contingent claim satisfies the partial differential equation

$$\begin{aligned}
0 &= \frac{1}{2}var(W)F_{WW} + \sum_{i=1}^{k} cov(W,Y_i)F_{WY_i} + \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{k} cov(Y_i,Y_j)F_{Y_iY_j} \\
&+ [r(W,Y,t)W - C^*(W,Y,t)]F_W \\
&+ \sum_{i=1}^{k} F_{Y_i}\Big[\mu_i - \Big(\frac{-J_{WW}}{J_W}\Big)(cov(W,Y_i) - \sum_{j=1}^{k}\Big(\frac{-J_{WY_j}}{J_W}\Big)(cov(Y_i,Y_j)\Big] \\
&+ F_t - r(W,Y,t)F + \delta(W,Y,t), \qquad (3.16)
\end{aligned}$$

where $r(W,Y,t)$ is given from equation (3.12).

The valuation equation (3.16) holds for any contingent claim. The boundary and appropriate terminal conditions are particular for each claim, which are defined by the particular contract. These conditions can be described by defining the price F on $[t,T) \times Z$, where $Z \subset (0,\infty) \times \mathbb{R}^k$ is an open set and ∂Z is its boundary. Furthermore, $\hat{\partial}Z$ is supposed to be the closed subset of ∂Z such that $(W(\tau),Y(\tau)) \in \hat{\partial}Z$ for all $(W(t),Y(t))$, where τ is the time of first passage from Z. That is, $\hat{\partial}Z$ is the set of all accessible boundary points. Consequently, (3.16) holds for all $(s,W(s),Y(s)) \in [t,T) \times Z$, with the contractual provisions determining the boundary information

$$\begin{aligned}
F(W(T),Y(T),T) &= \Theta(W(T),Y(T)), \quad W(T),Y(T) \in Z, \\
F(W(\tau),Y(\tau),\tau) &= \Psi(W(\tau),Y(\tau)), \quad W(\tau),Y(\tau) \in \hat{\partial}Z.
\end{aligned}$$

This result can be explained in a different way. The contingent claim F entitles its owner to receive three types of payments:

(1) if the underlying variables do not leave a certain region defined by Z before the maturity date T, a payment of Θ is received at the maturity date,

(2) if the underlying variables do leave the region before T, at time τ, and, therefore, belong to the set of all accessible boundary points, a payment of Ψ is received at that time, and

(3) a payout flow of δ is received until time T or time τ, whichever is sooner.

As a result, the unique solution of the partial differential equation can be written as:

$$
\begin{aligned}
F(W,Y,t,T) &= E_{W,Y,t}\Big[\Theta(W(T),Y(T)) && (3.17)\\
&\times \Big[e^{-\int_t^T \beta(W(u),Y(u),u)du}\Big] I(\tau \geq T) && (3.18)\\
&+ \Psi(W(\tau),Y(\tau),\tau) && (3.19)\\
&\times \Big[e^{-\int_t^T \beta(W(u),Y(u),u)du}\Big] I(\tau < T) && (3.20)\\
&+ \int_t^{\tau\wedge T} \delta(W(s),Y(s),s) && (3.21)\\
&\times \Big[e^{-\int_t^s \beta(W(u),Y(u),u)du}\Big] ds\Big], && (3.22)
\end{aligned}
$$

where E denotes expectation with respect to System I, $I(\cdot)$ is an indicator function, and τ is the time of first passage to $\hat{\partial}Z$.

3.2 A Theory of the Term Structure of Interest Rates (CIRII)

In this paper by Cox, Ingersoll, and Ross [6], the authors use the intertemporal general equilibrium asset pricing model presented in Section 3.1 to study the term structure of interest rates. The term structure of interest rates describes the relationship among the yields on default-free securities that differ only in their term to maturity. Therefore, it embodies the anticipations of the members of the market for future events. It could be used to examine the influence of changes in the underlying variables on the yield curve. In this model, the determinants of the term premiums are studied and how changes of variables will effect the term structure. However, it also considers ideas of several, well known, approaches to the determination of the term structure, like the *expectations hypothesis*, the *liquidity hypothesis*, and the *market segmentation hypothesis*.

The underlying general equilibrium model is modified and specialized to suit the needs of the described problem.

As shown in Section 3.1, the equilibrium interest rate satisfies (3.12). Moreover, it was proven that the equilibrium value of any contingent claim, F, satisfies the following differential equation:

$$\begin{aligned}
&\frac{1}{2}a^{*T}GG^Ta^*W^2F_{WW} + a^{*T}GS^TWF_{WY} + \frac{1}{2}tr(SS^TF_{YY}) \\
&\quad +(a^{*T}\alpha W - C^*)F_W + \mu^TF_Y + F_t + \delta - rF \\
&\quad = \phi_W F_W + \phi_Y F_Y,
\end{aligned} \tag{3.23}$$

where $\delta(W, Y, t)$ is the payout flow received by the security and

$$\begin{aligned}
\phi_W &= (a^{*T}\alpha - r)W \\
\phi_Y &= \Big(\frac{-J_{WW}}{J_W}\Big)a^{*T}GS^TW + \Big(\frac{-J_{WY}}{J_W}\Big)^T SS^T.
\end{aligned} \tag{3.24}$$

To apply these formulas to the problem of the term structure of interest rates, the authors specialize the preference structure first to the case of constant relative risk aversion utility functions and then further to the logarithmic utility function. In particular, they let $U(C(s), Y(s), s)$ be independent of the state variable Y and have the form

$$U(C(s), s) = e^{-\rho s}\Big[\frac{C(s)^\gamma - 1}{\gamma}\Big], \tag{3.25}$$

where ρ is a constant discount factor. It is easy to show that in this case the indirect utility function takes the form:

$$J(W, Y, t) \quad = \quad f(Y, t)U(W, t) + g(Y, t).$$

This special form brings about two important simplifications. First, the coefficient of relative risk aversion of the indirect utility function is constant, independent of both wealth and the state variables:[4]

$$\frac{-WJ_{WW}}{J_W} = 1 - \gamma. \tag{3.26}$$

Second, the elasticity of the marginal utility of wealth with respect to each of the state variables does not depend on wealth, and[5]

$$\frac{-J_{WY}}{J_W} = \frac{-f_Y}{f}. \tag{3.27}$$

As a result, a^* will depend on Y but not on W. Consequently, the vector of factor premiums, ϕ_Y, as defined in (3.24), also depends only on Y as does the

[4] For calculations see Appendix B.2

[5] For calculations see Appendix B.2.

equilibrium interest rate.

The logarithmic utility function corresponds to the special case of $\gamma = 0$.[6] As a result ϕ_Y reduces further to[7]

$$\phi_Y = a^{*T} GS. \tag{3.28}$$

In addition to that, a^* is explicitly given by

$$a^* = (GG^T)^{-1}\alpha + \Big(\frac{1 - 1^T(GG^T)^{-1}\alpha}{1^T(GG^T)^{-1}1}\Big)(GG^T)^{-1}1. \tag{3.29}$$

Since with constant relative risk aversions neither the interest rate r nor the factor risk premiums ϕ_Y depend on wealth, for such securities the partial derivatives F_W, F_{WW}, and F_{WY} are all equal to zero.

By combining these specializations, Cox, Ingersoll, and Ross find that the valuation equation (3.23) then reduces to

$$\frac{1}{2}tr(SS^T F_{YY}) + [\mu^T - a^{*T}GS^T]F_Y + F_t + \delta - rF = 0. \tag{3.30}$$

Furthermore, it is assumed that the change in production opportunities over time is described by a single state variable, $Y(\equiv Y_1)$. The development of the state variable Y is given by the stochastic differential equation

$$dY(t) = [\xi Y + \zeta]dt + \upsilon\sqrt{Y}d\omega(t), \tag{3.31}$$

with suitable variables.

Moreover, the means and variances of the rates of return on the production processes are proportional to Y. Consequently, it is convenient to introduce the notation $\alpha \equiv \hat{\alpha}Y$, $GG^T \equiv \Omega Y$, and $GS^T \equiv \Sigma Y$, where the elements of $\hat{\alpha}$, Ω, and Σ are constants.

As a result, the equilibrium interest rate can be written as:[8]

$$r(Y) = \Big(\frac{1^T\Omega^{-1}\hat{\alpha} - 1}{1^T\Omega^{-1}1}\Big)Y. \tag{3.32}$$

The interest rate thus follows a *diffusion process* with

$$drift(r) = \Big(\frac{1^T\Omega^{-1}\hat{\alpha} - 1}{1^T\Omega^{-1}1}\Big)(\xi Y + \zeta) \equiv \kappa(\theta - r) \tag{3.33}$$

$$var(r) = \Big(\frac{1^T\Omega^{-1}\hat{\alpha} - 1}{1^T\Omega^{-1}1}\Big)^2 \nu\nu^T Y \equiv \sigma^2 r. \tag{3.34}$$

[6]Note: $\lim_{\gamma\to 0} U(C(s), s) = ln(C(s))$.

[7]For calculations see Appendix B.2.

[8]For calculations see Appendix B.2.

It is convenient to define a new one-dimensional *Wiener process*, $z_1(t)$, such that:

$$\sigma\sqrt{r}dz_1(t) \equiv \nu\sqrt{Y}d\omega(t). \tag{3.35}$$

The interest rate dynamics can then be expressed as:

$$dr = \kappa(\theta - r)dt + \sigma\sqrt{r}dz_1. \tag{3.36}$$

The long-term value is θ and κ determines the speed of adjustment. Expression (3.36) corresponds to a continuous time first-order autoregressive process, where the randomly moving interest rate is elastically pulled toward θ if $\kappa > 0$ and $\theta > 0$. This structure leads to an interest rate behavior, which is empirically relevant. Negative spot rates are precluded and if the spot rate reaches zero, it can subsequently become positive. The absolute variance increases with an increase in r. The probability density of the interest rate at time s, conditional on its value at the current time t, is given by:

$$f(r(s), s; r(t), t) = ce^{-u-v}\left(\frac{v}{u}\right)^{\frac{q}{2}} I_q(2(uv)^{\frac{1}{2}}), \tag{3.37}$$

where

$$c \equiv \frac{2\kappa}{\sigma^2(1 - e^{-\kappa(s-t)})} \tag{3.38}$$

$$u \equiv cr(t)e^{-\kappa(s-t)} \tag{3.39}$$

$$v \equiv cr(s) \tag{3.40}$$

$$q \equiv \frac{2\kappa\theta}{\sigma^2} - 1 \tag{3.41}$$

and $I_q(\cdot)$ is the *modified Bessel function of the first kind of order* q. The distribution function is the *noncentral chi-square*, $\chi^2[2cr(s); 2q + 2, 2u]$, with $2q + 2$ *degrees of freedom* and *parameter of noncentrality* $2u$ proportional to the current spot rate.

Straightforward calculations give the expected value and variance of $r(s)$ as:[9]

$$E(r(s) \mid r(t)) = r(t)e^{-\kappa(s-t)} + \theta(1 - e^{-\kappa(s-t)}) \tag{3.42}$$

$$var(r(s) \mid r(t)) = r(t)\left(\frac{\sigma^2}{\kappa}\right)(e^{-\kappa(s-t)} - e^{-2\kappa(s-t)}) \tag{3.43}$$

$$+\theta\left(\frac{\sigma^2}{2\kappa}\right)(1 - e^{-\kappa(s-t)})^2. \tag{3.44}$$

[9]For calculations see Appendix B.2.

Note that the instantaneous interest rate is proportional to the state variable and thus can be thought of as the state variable itself.

Consider the problem of valuing a default-free discount bond promising to pay one unit at time T. The prices of these bonds for all T will completely determine the term structure. Under these assumptions, the factor risk premium in (3.24) is[10]

$$\Big[\hat{\alpha}^T\Omega^{-1}\Sigma + \Big(\frac{1 - 1^T\Omega^{-1}\hat{\alpha}}{1^T\Omega^{-1}1}\Big)1^T\Omega^{-1}\Sigma\Big]Y \equiv \lambda Y. \tag{3.45}$$

By using (3.45) and (3.33), one can write the fundamental equation (3.30) for the price of a discount bond, P, most conveniently as

$$\frac{1}{2}\sigma^2 r P_{rr} + \kappa(\theta - r)P_r + P_t - \lambda r P_r - rP = 0 \tag{3.46}$$

with boundary condition $P(r,T,T) = 1$. The first three terms in (3.46) are the expected price change for the bond. As the rate of return can be written as $\Delta P/P$, the expected rate of return on the bond is $r + (\lambda r P_r/P)$. λr is the covariance of changes in the interest rate with percentage changes in optimally invested wealth (market portfolio). The absence of arbitrage means that the return/risk ratio should be the same for all assets. λ is the market value of risk.[11]

By taking the relevant expectation, we obtain the bond prices as:

$$P(r,t,T) = A(t,T)e^{-B(t,T)r}, \tag{3.47}$$

where

$$A(t,T) \equiv \Big(\frac{2\gamma e^{\frac{(\kappa+\lambda+\gamma)(T-t)}{2}}}{(\gamma+\kappa+\lambda)(e^{\gamma(T-t)}-1)+2\gamma}\Big)^{\frac{2\kappa\theta}{\sigma^2}} \tag{3.48}$$

$$B(t,T) \equiv \frac{2(e^{\gamma(T-t)}-1)}{(\gamma+\kappa+\lambda)(e^{\gamma(T-t)}-1)+2\gamma} \tag{3.49}$$

$$\gamma \equiv ((\kappa+\lambda)^2 + 2\sigma^2)^{\frac{1}{2}}. \tag{3.50}$$

Bonds are commonly quoted in terms of yields rather than prices. For the discount bonds we are now considering the yield-to-maturity $R(r,t,T)$ is defined by:

$$\begin{aligned} P(r,t,T) &= e^{-(T-t)R(r,t,T)} \\ R(r,t,T) &= \frac{rB(t,T) - ln(A(t,T))}{(T-t)}. \end{aligned} \tag{3.51}$$

[10]For calculations see Appendix B.2.

[11]If one considers the no-arbitrage condition, one can formulate a pricing kernel. The diffusion

coefficient of the dynamics is restricted to be the negative of λ.

As maturity nears, the yield-to-maturity approaches the current interest rate independently of any of the parameters. As we consider longer and longer maturities, the yield approaches a limit which is independent of the current interest rate:

$$Rl = R(r, t, \infty) = \frac{2\kappa\theta}{\gamma + \kappa + \lambda}. \tag{3.52}$$

As one can see in Section 3.3, the value of

$$Rg = \frac{\kappa\theta}{\kappa + \lambda} \tag{3.53}$$

plays an important role determining the appearance of the term structure.

There are certain conditions on the variables, which can be written as follows:

$$\kappa\theta \geq 0 \tag{3.54}$$
$$\sigma^2 > 0 \tag{3.55}$$
$$\lambda < 0 \tag{3.56}$$
$$\kappa > 0 \tag{3.57}$$
$$\theta > 0 \tag{3.58}$$
$$\kappa > |\lambda| \tag{3.59}$$

The condition on λ ensures positive premiums, since $P_r < 0$. Furthermore, the condition $\kappa > |\lambda|$ ensures a positive value for (3.53).

In the paper of Brown and Dybvig [3] new variables are introduced:[12]

$$\phi_1 = [(\kappa + \lambda)^2 + 2\sigma^2]^{\frac{1}{2}} \tag{3.60}$$
$$\phi_2 = \frac{\kappa + \lambda + \phi_1}{2} \tag{3.61}$$
$$\phi_3 = \frac{2\kappa\theta}{\sigma^2}. \tag{3.62}$$
$$\tag{3.63}$$

These offer certain advantages when testing the implications of that model.

Using these variables the equations (3.48)-(3.49) can be written as:

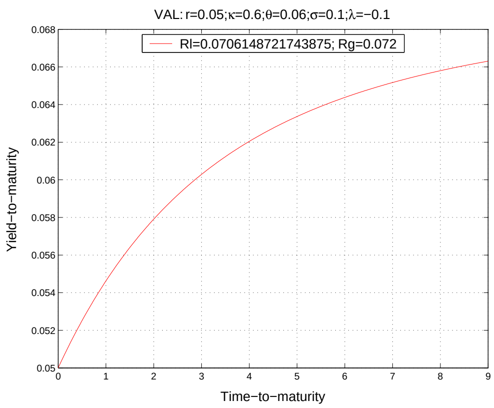
$$A(t,T) \equiv \left(\frac{\phi_1 e^{\phi_2(T-t)}}{\phi_2(e^{\phi_1(T-t)} - 1) + \phi_1}\right)^{\phi_3} \tag{3.64}$$
$$B(t,T) \equiv \left(\frac{e^{\phi_1(T-t)} - 1}{\phi_2(e^{\phi_1(T-t)} - 1) + \phi_1}\right). \tag{3.65}$$

[12] I used these variables in all MATLAB files.

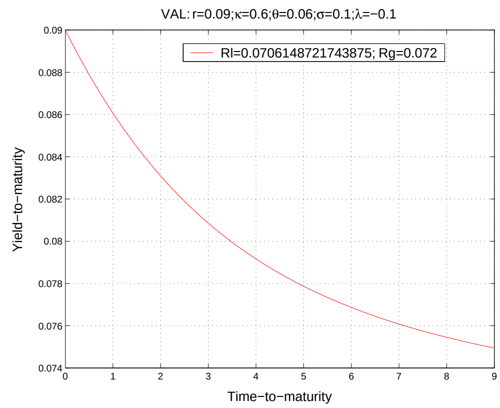
3.3 Characteristics of the Term Structure

In this section, characteristics of the term structure evolved by the model of Cox, Ingersoll, and Ross [6] are described and partially visualized.[13] The authors show that the term structure is uniformly rising as long as the spot rate is below the long-term yield (3.52). Secondly, with an interest rate in excess of (3.53), the term structure is falling. Finally, for intermediate values of the interest rate, the yield curve is humped. Figure 3.1 displays the results stated above for certain values of the variables. The values chosen to visualize the characteristics satisfy the conditions (3.54)-(3.59).

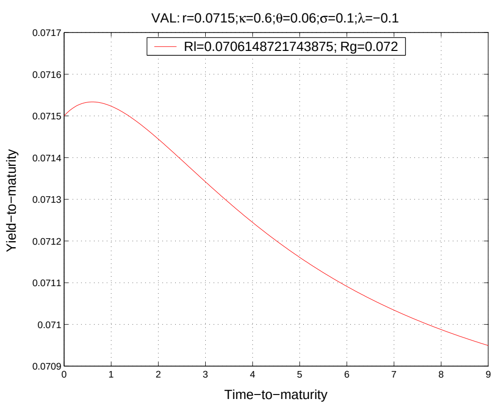
[13] For MATLAB source file 'characterize.m' and 'termstructure.m' see Appendix C.



(a) $r < \frac{2\kappa\theta}{\gamma+\kappa+\lambda}$



(b) $r > \frac{\kappa\theta}{\kappa+\lambda}$



(c) $\frac{2\kappa\theta}{\gamma+\kappa+\lambda} < r < \frac{\kappa\theta}{\kappa+\lambda}$

Figure 3.1: Term Structure dependency on the spot rate

Moreover, several other comparative statics for the yield curve are easily obtained. The reader may take into consideration that any change of the variables changes the value of (3.52) and possibly changes the value of (3.53).

In the paper it is shown that an increase in the current interest rate increases the yields for all maturities, but the effect is greater for shorter maturities. This can be seen if one considers that a bond's yield is a composition of the spot rate and a premium. As a consequence, a higher spot rate increases all yields. The long-term value of the spot rate, θ, has not changed. Hence, the effect is greater for shorter maturities.

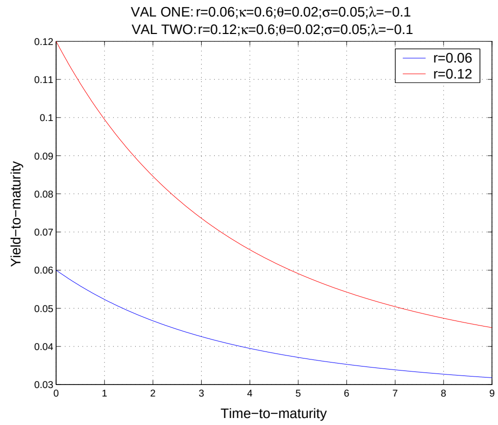


Figure 3.2: Effect of an increase in r on the term structure

Similarly, an increase in the steady state mean θ increases all yields, but here the effect is greater for longer maturities as the long-term value θ has changed, whereas the current interest rate r has not.

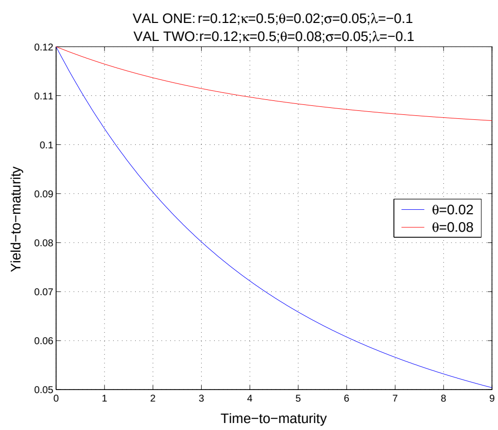


Figure 3.3: Effect of an increase in θ on the term structure

The yields-to-maturity decrease as σ^2 or λ increases. This can be easily seen as one remembers that higher values of λ indicate lower premiums as λ is the market value of risk. As λ increases (or $|\lambda|$ decreases) the value of risk decreases. Higher values of the variance of the interest rate, σ^2, indicates more uncertainty about future real production opportunities, and thus more uncertainty about future consumption. As a consequence, the guaranteed claim in a bond is valued more highly by investors.

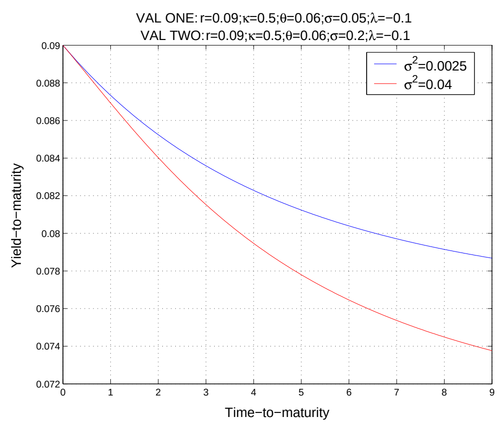


Figure 3.4: Effect of an increase in σ^2 on the term structure

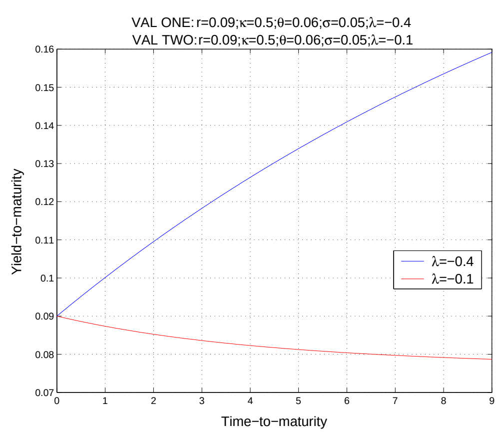
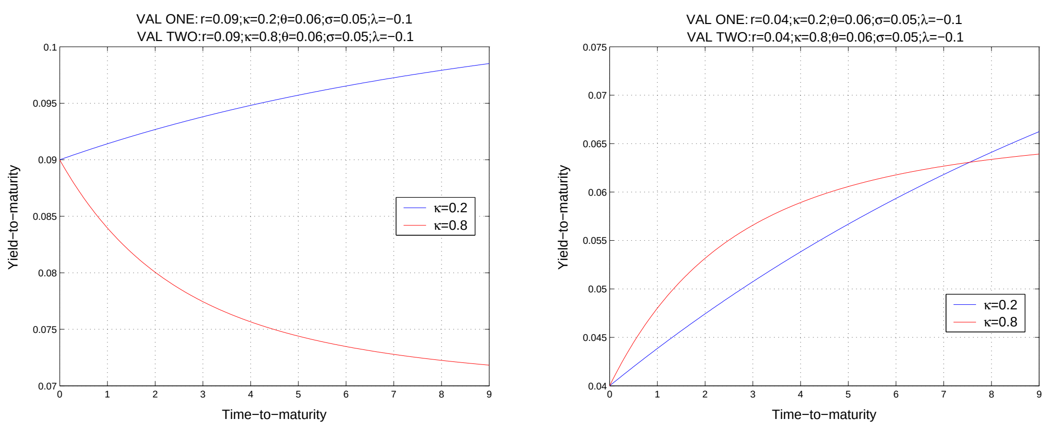


Figure 3.5: Effect of an increase in λ on the term structure

The effect of a change in κ may be of either sign depending on the current interest rate, that is, the yield is an increasing function of the speed of adjustment parameter κ if the interest rate is less than θ and a decreasing function of κ if the interest rate is greater than θ, respectively. This can be seen if one considers that a higher value of κ means that the spot rate adjusts faster to a higher/lower level.



(a) $r >= \theta$

(b) $r < \theta$

Figure 3.6: Effect of an increase in κ on the term structure

Chapter 4

The Expected Exchange and Depreciation Rates

In this chapter, expectations of the future exchange rates are formed.

4.1 The Underlying Economic Model

The main issue under discussion is how the expected exchange rate of two currencies may evolve and what characteristics may be observed under certain circumstances. I will not confine myself to the analyses of two particular countries, e.g. Germany and the United States of America. Furthermore, I will not distinguish between relatively small and big countries, as the price level and the countries' influence on the price will not appear in my model.

In my approach, two countries, country X and country Y, with different currencies are investigated. Note that all variables used to describe the economy of country Y are indicated by *. When describing the exchange rate or the expectation of the future exchange rate, the price quotation is used from country X's point of view. The investigations will be kept as general as possible such that the results may be specialized to investigate particular problems.

In order to describe the structure and the characteristics of the expected exchange rates of those two currencies, I applied the model of Cox, Ingersoll, and Ross for modelling the term structure to each of the two countries (see Section 3.2). It is possible to examine various situations as I only restricted the variables describing the economies to certain reasonable intervals, but not to certain values. There is no trading of goods between the countries. As a result, the theory of the purchasing power parity is neglected in this approach. The only way the expectations of the future exchange rate appreciation and depreciation, respectively, are evolved is that of different investment opportunities in bonds. That is, the investors observe the term structure of their own country and that of the foreign country. Differences in yields-to-maturity are seen as the reason for changes in the exchange rate in the future to ensure that the uncovered interest rate parity is valid.

Another approach could have been to extend the model of Cox, Ingersoll, and Ross [6] to be an open economy, e.g. by introducing state variables describing the foreign country. I did not pursue that approach. However, I would like to point out that there are a few approaches of extending the model, or similar models, to be an open economy. For example, refer to Pavlova and Rigobon [14].

First of all, the idea of the interest rate parity is introduced.

4.2 The Interest Rate Parity

The interest rate parity is a relationship that must hold between the spot interest rates of two currencies and the current and expected exchange rate if there are to be no arbitrage opportunities. Considering the opportunity of investing in bonds either of country X or country Y leads to the idea of the equality of yields of both opportunities. An investment of amount x in the country X, valued in the currency of country X, results in an amount of $(1+i)x$ after one period if an interest rate of i is paid. If you consider an exchange rate of $\mathcal{E}_t$ at time t, the amount x is equal to an amount $\frac{x}{\mathcal{E}_t}$ valued in the currency of country Y. At the end of the period the exchange rate is expected to be of a particular value such that the yield is as high as in country X. Otherwise, an inequality would lead to arbitrage opportunities and, consequently, to an appreciation or depreciation of the current exchange rate respectively.

This equality can be written as:

$$(1+i) \quad = \quad (1+i^*)\frac{\mathcal{E}^e_{t+1}}{\mathcal{E}_t}, \tag{4.1}$$

where $\mathcal{E}^e_{t+1}$ stands for the expected value of the exchange rate at time $t+1$.

Using the fact that $ln(1+i) \approx i$ for small i leads to the following approximation:

$$\begin{aligned}
(1+i) &= (1+i^*)\frac{\mathcal{E}^e_{t+1}}{\mathcal{E}_t} \\
(1+i) &= (1+i^*)(1+\frac{\mathcal{E}^e_{t+1}-\mathcal{E}_t}{\mathcal{E}_t}) \\
ln(1+i) &= ln(1+i^*)+ln(1+\frac{\mathcal{E}^e_{t+1}-\mathcal{E}_t}{\mathcal{E}_t}) \\
\Rightarrow i &\approx i^*+\frac{\mathcal{E}^e_{t+1}-\mathcal{E}_t}{\mathcal{E}_t}. \qquad (4.2)
\end{aligned}$$

This is commonly known as the *uncovered* interest rate parity.

On the other hand, the *covered* interest rate parity provides an expression for the forward premium of discount that merchants or investors would have to pay to hedge or cover the exchange rate risk associated with a contract to receive or deliver foreign currency in the future. In my thesis, the research done is based on the uncovered interest rate parity.

4.3 The Expected Exchange Rates

The uncovered interest rate parity can be used to infer the entire path of expected future values of the spot exchange rate given the observed term structure of domestic and foreign interest rates and the spot exchange rate. As the agents expect the assumptions of the interest rate parity to be fulfilled eventually, the expectation of the future spot exchange rate at any future time s is formed according to the parity condition. As the model assumes continuous payments of interest, it is important to distinguish between the yield-to-maturity $R(r,t,T)$ and the total return (TR), which can be written as:

$$R(r,t,T)^{TR} = e^{R(r,t,T)\cdot(T-t)} - 1, \tag{4.3}$$

where $T-t$ stands for the time-to-maturity. It is obvious that the continuous payment of interest results in higher capital at maturity than a discrete payment of interest. It can be easily shown[1] that (4.3) is equivalent to

$$R(r,t,T)^{TR} = \frac{1}{P(r,t,T)} - 1. \tag{4.4}$$

Taking that into account, (4.1) can be used to calculate the expectation of the exchange rate at time T as follows:

$$\begin{aligned}
\mathcal{E}_T^e &= \frac{1 + R(r,t,T)^{TR}}{1 + R^*(r^*,t,T)^{TR}}\mathcal{E}_t & (4.5)\\
&= \frac{\frac{1}{P(r,t,T)}}{\frac{1}{P^*(r^*,t,T)}}\mathcal{E}_t & (4.6)\\
&= \frac{P^*(r^*,t,T)}{P(r,t,T)}\mathcal{E}_t. & (4.7)
\end{aligned}$$

Using the approximation (4.2) mentioned above leads to an expected exchange rate of

$$\mathcal{E}_T^e = (R(r,t,T)^{TR} - R^*(r^*,t,T)^{TR})\mathcal{E}_t + \mathcal{E}_t. \tag{4.8}$$

In the following, (4.5) is implemented, because the errors resulting from the fact that (4.8) is only an approximation may not be ignored if longer maturities are investigated.

4.4 The Expected Depreciation Rates

The expectation regarding the exchange rate at any future time T can be used to calculate the expected depreciation rate. Taking the values of the expected exchange rates at any arbitrary time j, $\mathcal{E}_j^e$, and i, $\mathcal{E}_i^e$, with $j > i$, the depreciation rate, which is expected by a representative agent, can be written as follows:

$$\frac{\mathcal{E}_j^e - \mathcal{E}_i^e}{\mathcal{E}_i^e}. \tag{4.9}$$

Positive values of (4.9) represent an expected depreciation of the exchange rate. Negative values indicate an appreciation. Moreover, especially the expected depreciation rate within one period can be estimated using (4.9). If the length of one period equals Δ, the expected depreciation rate at time s can be written as:

$$\Big(\frac{\mathcal{E}_{s+\Delta}^e - \mathcal{E}_s^e}{\mathcal{E}_s^e}\Big)_1, \tag{4.10}$$

where the index $(\cdot)_1$ is introduced for later purposes.

[1] For calculations see Appendix B.3.

Chapter 5

The Expected Future One Period Total Returns (ExpTR(Δ))

In this chapter, the attention is paid on the *expected future one period total return* (ExpTR(Δ)), that is, the value of

$$E(R(r(s), s, s + \Delta)^{TR}).$$

Consequently, it deals with the following question: What does an individual of this economy expect at the current time t of the total return of a bond with time-to-maturity of one period at a future time s? Analogous to that, the ExpTR(Δ)* can be examined. Especially the development of the expected *difference* of the total return of a bond of country X and a bond of country Y with time-to-maturity of one period at time s (DiffExpTR(Δ)) , denoted by

$$\begin{aligned} E(R(r(s), s, s + \Delta)^{TR} - R^*(r^*(s), s, s + \Delta)^{TR}) &= \\ = E(R(r(s), s, s + \Delta)^{TR}) - E(R^*(r^*(s), s, s + \Delta)^{TR}), \end{aligned} \tag{5.1}$$

plays an important role in Section 5.3. $E(\cdot)$ stands for the expectation. Note that for $s = t$ the expectation is already known. It is equivalent to the total return of a bond with time-to-maturity $T - t = \Delta$, which is known from the term structure. When dealing with the expectation, therefore, one can assume $s > t$. This approach will lead to another expectation of future exchange rates and depreciation rates respectively.

The abbreviations ExpTR(Δ) and DiffExpTR(Δ) are also used to describe the expected future one period total returns and the differences of the expected total returns at *all future times* respectively. The particular meaning of ExpTR(Δ) and DiffExpTR(Δ), respectively, will be clear out of the used *context*.

First of all, however, the *expectations hypothesis* and the *liquidity preference hypothesis* need to be introduced.

5.1 The Expectations Hypothesis

The basic idea is that, with the exception of a term premium, there should be no expected difference in return from holding a long-term bond or rolling over a sequence of short-term bonds. An equivalent statement is that the expected holding period return on short-term bonds equals the expected holding period return on long-term bonds. This hypothesis assumes competitive markets, individuals maximizing their expected profit by investing in default free bonds, and the lack of transaction costs. As a result, a lasting difference between yields of a long-term bond and of rolling over a sequence of short-term bonds cannot be observed, because arbitrage would lead to an adjustment of prices and yields, respectively, and, therefore, to an equilibrium. Consequently, the expectations hypothesis can be described by the assumption that, in equilibrium, long-term yields are geometric averages[1] of current and expected future short-term yields.

Knowledge of the term structure of interest rates of an economy combined with the assumptions of the expectations hypothesis can be used to evolve expectations of the future short-term interest rates. Furthermore, under the simple, risk-neutral *efficient markets hypothesis*, the forward rate, which is an implicit interest rate that is valid today for a contract in the future, is the optimum predictor of the future spot rate as the prices of assets reflect all available information.

However, there is now a consensus within the profession that the simple, risk-neutral efficient markets hypothesis has been decisively rejected. Various empirical investigations have shown a difference between the forward rates and the actual spot rates, especially, if longer maturities are investigated. This may have

[1]The geometric average of n positive variables $a_1, a_2, \ldots, a_n$ is defined as $\sqrt[n]{a_1 a_2 \ldots a_n}$.

several reasons. For example, the choice of assumptions may not be a suitable description of real behavior of the agents and of given facts of the market. For further information refer to Gischer, Herz and Menkhoff [8].

If we allow for some nonrational behavior of financial markets, various other hypotheses explaining the term structure could be mentioned. In my thesis, the liquidity preference hypothesis plays an important role, as a certain preference for holding money instead of other forms of wealth is assumed.

5.2 The Liquidity Preference Hypothesis

According to the expectations hypothesis, there should be no expected difference in return from holding a long-term bond or rolling over a sequence of short-term bonds. However, people want to hold money for the purpose of making everyday market purchases. This explains the transactions demand. Furthermore, people hold money for sudden emergency purchases and unexpected market transactions as well as for speculative purposes and later financial opportunities. These two facts result in a precautionary and speculative demand. The desire, however, to hold money rather than other forms of wealth, which may result from the transactions demand, the precautionary demand and the speculative demand, lead to an inequality of the return from a long-term bond and the return stemming from a sequence of short-term bonds.

Several possibilities exist to define what exactly a liquidity preference means. First of all, one could say that the return of bonds with longer maturities is always higher than the return from investing in bonds with shorter time-to-maturity repeatedly, that is, the inequality is assumed to exist at any time. However, I only assume that the return of bonds with a long time-to-maturity is only higher than the return from investing in bonds with shorter time-to-maturity repeatedly if the maturity date is not in the relatively near future, but in the sufficiently distant future. Hence, the model is less restricted.

The liquidity preference, as defined above, can be fulfilled by assuming $R(r,t,\infty) > \theta$. This can be seen if one assumes that the expected future one period total return at time s can be approximated by $e^{E(r(s)|r(t))B(s,s-t)-ln(A(s,s-t))}-1$, where $E(r(s)|r(t))$ is calculated as in (3.42). As stated in Section 3.2, the yield (3.51) approaches the spot rate as maturity nears. Consequently, the assumption of a liquidity preference for bonds with a maturity date in the sufficiently distant future can now be written as:

$$e^{r(t)\Delta}e^{E(r(t+\Delta)|r(t))\Delta}\cdots e^{E(r(T)|r(t))\Delta} \quad < \quad e^{R(r,t,T)(T-t)}. \tag{5.2}$$

If the maturity date is in the sufficiently distant future, the yield-to-maturity $R(r,t,T)$ is approximately $R(r,t,\infty)$ and, considering (3.42), the expected spot rate given the current interest rate $E(r(T)|r(t))$ is approximately θ. If the inequation $R(r,t,\infty) > \theta$ is valid, there is a date of maturity T such that inequation (5.2) holds.

Although $e^{E(r(s)|r(t))B(s,s-t)-ln(A(s,s-t))} - 1$ is not the correct expected future one period total return as

$$E\big(e^{r(s)B(s,s-t)-ln(A(s,s-t))}\big) \;\; \neq \;\; e^{E(r(s)|r(t))B(s,s-t)-ln(A(s,s-t))},$$

it can be shown that for a sufficiently short length of one period the approximation

$$E\big(e^{R(r(s),s,s+\Delta)\Delta}\big) \approx e^{E(r(s)|r(t))\Delta}, \tag{5.3}$$

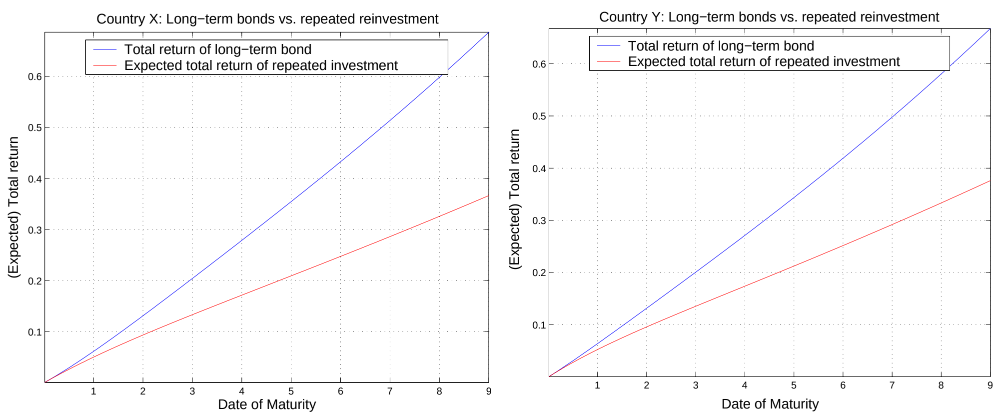
can be proven valid.[2] Consequently, the condition $R(r,t,\infty) > \theta$ also ensures a liquidity preference if the correct expected future one period total returns are investigated.

Analogous to that, a liquidity preference in country Y can be ensured by assuming $R^*(r^*,t,\infty) > \theta^*$.

All examples presented in this thesis (see Chapter 7), where the length of the period Δ is arbitrarily set to 0.01, support this assumption.

[2] For proof see Appendix B.4.

The following figure visualizes the idea of a liquidity preference for a certain composition of the variables.[3]



(a) Country X

(b) Country Y

Figure 5.1: Geometric average of short term yields vs. long-term yield

As one can see, the yield-to-maturity of long-term bonds is higher than the yield stemming from a repeated investment in short-term bonds.

5.3 More Expected Depreciation Rates

In comparison with (4.10), the ExpTR(Δ) and the ExpTR(Δ)* can be used to calculate another expectation of the one period depreciation rate.

By the uncovered interest rate parity condition the expected rate of exchange rate depreciation is just equal to the relevant interest rate differential. That is, the expected exchange rate depreciation is equal to the interest differential on financial assets in the relevant currencies with the same maturity and identical risk characteristics. As a result, using the assumption of investing an amount of x of currency of country X one can write the following equation:

$$x(1 + E(R(r(s), s, s + \Delta)^{TR})) \quad = \quad x(1 + E(R^*(r^*(s), s, s + \Delta)^{TR}))\frac{\mathcal{E}^e_{s+\Delta}}{\mathcal{E}^e_s}$$

[3]For MATLAB source file 'testliqpref.m' see Appendix C.

$$\Rightarrow \quad \Big(\frac{\mathcal{E}^e_{s+\Delta} - \mathcal{E}^e_s}{\mathcal{E}^e_s}\Big)_2 \quad = \quad \frac{\frac{1+E(R(r(s),s,s+\Delta)^{TR})}{1+E(R^*(r^*(s),s,s+\Delta)^{TR})}\mathcal{E}^e_s - \mathcal{E}^e_s}{\mathcal{E}^e_s}$$

$$\begin{aligned} &= \frac{1 + E(R(r(s), s, s+\Delta)^{TR})}{1 + E(R^*(r^*(s), s, s+\Delta)^{TR})} - 1 \\ &= \frac{1 + E(R(r(s), s, s+\Delta)^{TR}) - 1 - E(R^*(r^*(s), s, s+\Delta)^{TR})}{1 + E(R^*(r^*(s), s, s+\Delta)^{TR})} \\ &= \frac{E(R(r(s), s, s+\Delta)^{TR}) - E(R^*(r^*(s), s, s+\Delta)^{TR})}{1 + E(R^*(r^*(s), s, s+\Delta)^{TR})}, \end{aligned} \tag{5.4}$$

where the index $(\cdot)_2$ is introduced for later purposes. As one can see, the DiffExpTR(Δ) determines the expected depreciation rate. Note that for any time s the expected depreciation rate can be calculated using (5.4). According to the simple, risk-neutral efficient markets hypothesis mentioned in Section 5.1, the relevant interest rate differential is the optimum predictor of the future exchange rate depreciation.

The two different expected depreciation rates merit closer examination. This poses the question of whether these two expectations are different, show a similar structure or are even the same.

The expectations hypothesis stated in Section 5.1 leads to the logical conclusion that both expected depreciation rates need to be the same, that is, the expectations hypothesis can be used to formulate an equilibrium between the expected one period depreciation rate of the exchange rate given the term structure and the expected depreciation rate given the DiffExpTR(Δ) of bonds with time-to-maturity of Δ. This can be easily seen if one considers the opportunity of investing in either bonds in country X or in bonds in country Y. The expectations hypothesis states that the return from holding a long-term bond, e.g. with time-to-maturity $s + \Delta - t$, is the same as rolling over a sequence of short-term bonds (here short-term is equivalent to one period). Consequently, the expected exchange rate at time $s + \Delta - t$ is the same if one holds a long-term bond with time-to-maturity $s+\Delta-t$ or reinvests in short-term bonds repeatedly. The same argument is assumed to be valid for a long-term bond with time-to-maturity $s-t$. Hence, the calculated depreciation rate using the term structure needs to be the same as the depreciation rate resulting from the DiffExpTR(Δ) at time s, such that:

$$\Big(\frac{\mathcal{E}^e_{s+\Delta} - \mathcal{E}^e_s}{\mathcal{E}^e_s}\Big)_1 = \Big(\frac{\mathcal{E}^e_{s+\Delta} - \mathcal{E}^e_s}{\mathcal{E}^e_s}\Big)_2. \tag{5.5}$$

In other words, the depreciation rate calculated using $R(r(t), t, s + \Delta)$ and $R^*(r^*(t), t, s + \Delta)$ is the same as the one using the ExpTR(Δ) and ExpTR(Δ)* at time s respectively.

Considering the liquidity preference, however, the assumed equality (5.5) of the expected depreciation rates given the expected exchange rate stemming from the term structure and the expected depreciation rates based on the expectation regarding the one period total return differentials may not be valid. This follows from the fact that the equality of yields assumed by the expectations hypothesis is not valid if a liquidity preference exists.

In Chapter 7, the structure of the two different expected depreciation rates of the various examples will be investigated. It will become obvious that neither an absolutely similar nor an absolutely different structure of the depreciation rates can be observed, independently from the choice of the factors. Moreover, in some examples a similar development can be observed, whereas in other examples the paths differ from each other significantly.

Moreover, I would like to point out that the ExpTR(Δ) and the ExpTR(Δ)* could also be used to calculate the expected exchange rate instead of the expected depreciation rate. However, I confine myself to the investigation to the expected depreciation rate. The investigation of the differences between the expectations stemming from the term structure and those stemming from the ExpTR(Δ) and the ExpTR(Δ)* would not broaden our knowledge.

5.4 Calculation of the ExpTR(Δ)

In order to investigate the characteristics of the DiffExpTR(Δ) mentioned above (see (5.1)), the expectation needs to be given explicitly in a form dependent on the variables which may influence the behavior.[4] Basically, an explicit form of

$$E(R(r(s), s, s+\Delta)^{TR}) = \tag{5.6}$$

$$\begin{aligned} &= E(e^{R(r(s),s,s+\Delta)\Delta} - 1) \\ &= E(e^{r(s)B(s,s+\Delta) - ln(A(s,s+\Delta))}) - 1 \end{aligned}$$

for a particular future time s is needed.

Note that $A(s, s+\Delta)$ and $B(s, s+\Delta)$ are defined as in (3.48) and (3.49) respectively. If one takes a closer look at those definitions, it is obvious that $A(s, s+\Delta)$ and $B(s, s+\Delta)$ are constant for any choice of s and independent from the current time t and the spot rate $r(t)$. For an easier notation, $A(s, s+\Delta)$ and $B(s, s+\Delta)$ are replaced by $\bar{A}$ and $\bar{B}$ respectively. The variable c is defined as

[4] For a possible alternative calculation see Appendix B.5.

in (3.38). The current interest rate $r(t)$ influences the value of u, which is defined as in (3.39).

In Chapter 3, the probability density (3.37) of the interest rate at time s, $r(s)$, conditional on its value at the current time t, was introduced. According to the economic model, negative interest rates are excluded. Furthermore, using the fact that the distribution function is noncentral chi-square with $2q+2$ degrees of freedom and a parameter of noncentrality of $2u$, where q is defined as in (3.41), the explicit form of the expectation can be derived. Note, that v is defined as in (3.40).

First of all, the probability density of a distribution function, which is noncentral chi-square with n degrees of freedom and a parameter of noncentrality of $\lambda > 0$, needs to satisfy the following condition:[5]

$$\int_0^\infty \frac{1}{2} e^{-\frac{x+\lambda}{2}} \left(\frac{x}{\lambda}\right)^{\frac{n-2}{4}} I_{\frac{n-2}{2}}((\lambda x)^{\frac{1}{2}})dx = 1. \tag{5.7}$$

With

$$\begin{aligned} \bar{v} &= v - r(s)\bar{B} \\ &= cr(s) - r(s)\bar{B} \\ &= r(s)(c - \bar{B}) \qquad (5.8) \\ \bar{u} &= \frac{uv}{\bar{v}} \\ &= \frac{ucr(s)}{r(s)(c-\bar{B})} \\ &= \frac{uc}{c-\bar{B}}, \qquad (5.9) \end{aligned}$$

where $\bar{u} > 0$[6], and (3.37), the expectation of $e^{r(s)\bar{B}-ln(\bar{A})}$ can be written as follows:

$$E(e^{r(s)\bar{B}-ln(\bar{A})}) = \tag{5.10}$$

$$\begin{aligned} &= \int_0^\infty e^{r(s)\bar{B}-ln(\bar{A})} c e^{-u-v} \left(\frac{v}{u}\right)^{\frac{q}{2}} I_q(2(uv)^{\frac{1}{2}})dr(s) \\ &= \int_0^\infty \frac{c}{\bar{A}} e^{r(s)\bar{B}-u-v} \left(\frac{v}{u}\right)^{\frac{q}{2}} I_q(2(\bar{u}\bar{v})^{\frac{1}{2}})dr(s) \\ &= \int_0^\infty \frac{c}{\bar{A}} e^{\bar{u}-\bar{u}} e^{-u-\bar{v}} \left(\frac{v}{u}\right)^{\frac{q}{2}} I_q(2(\bar{u}\bar{v})^{\frac{1}{2}})dr(s) \end{aligned}$$

[5] For description of the noncentral chi-square see Mueller [12].

[6] For further details see Appendix B.5.

$$\begin{aligned}
&= \int_0^\infty \frac{c}{\bar{A}} e^{\bar{u}-u} e^{-\bar{u}-\bar{v}} \left(\frac{v}{u}\right)^{\frac{q}{2}} I_q(2(\bar{u}\bar{v})^{\frac{1}{2}}) dr(s) \\
&= \int_0^\infty \frac{c}{\bar{A}} \left(\frac{1}{c}\right)^{-q} \left(\frac{1}{c}\right)^{q} (c-\bar{B})^{-q} (c-\bar{B})^{q} e^{\bar{u}-u} e^{-\bar{u}-\bar{v}} \left(\frac{v}{u}\right)^{\frac{q}{2}} I_q(2(\bar{u}\bar{v})^{\frac{1}{2}}) dr(s) \\
&= \int_0^\infty \frac{c}{\bar{A}} \left(\frac{1}{c}\right)^{-q} (c-\bar{B})^{-q} e^{\bar{u}-u} e^{-\bar{u}-\bar{v}} \left(\frac{v(c-\bar{B})^2}{uc^2}\right)^{\frac{q}{2}} I_q(2(\bar{u}\bar{v})^{\frac{1}{2}}) dr(s) \\
&= \int_0^\infty \frac{c}{\bar{A}} \left(\frac{1}{c}\right)^{-q} (c-\bar{B})^{-q} e^{\bar{u}-u} e^{-\bar{u}-\bar{v}} \left(\frac{cr(s)(c-\bar{B})^2}{uc^2}\right)^{\frac{q}{2}} I_q(2(\bar{u}\bar{v})^{\frac{1}{2}}) dr(s) \\
&= \int_0^\infty \frac{c}{\bar{A}} \left(\frac{1}{c}\right)^{-q} (c-\bar{B})^{-q} e^{\bar{u}-u} e^{-\bar{u}-\bar{v}} \left(\frac{r(s)(c-\bar{B})}{\frac{uc}{c-\bar{B}}}\right)^{\frac{q}{2}} I_q(2(\bar{u}\bar{v})^{\frac{1}{2}}) dr(s) \\
&= \int_0^\infty \frac{c}{\bar{A}} \left(\frac{c-\bar{B}}{c}\right)^{-q} e^{\bar{u}-u} e^{-\bar{u}-\bar{v}} \left(\frac{\bar{v}}{\bar{u}}\right)^{\frac{q}{2}} I_q(2(\bar{u}\bar{v})^{\frac{1}{2}}) dr(s). \qquad (5.11)
\end{aligned}$$

Using the substitution

$$\begin{aligned}
n &= 2q+2 \qquad (5.12) \\
\lambda &= 2\bar{u} \qquad (5.13) \\
\bar{v} &= \frac{x}{2} \qquad (5.14)
\end{aligned}$$

and considering the chain rule of differentiation, (5.7) is equivalent to

$$1 = \int_0^\infty \frac{1}{2} e^{-\frac{x+\lambda}{2}} \left(\frac{x}{\lambda}\right)^{\frac{n-2}{4}} I_{\frac{n-2}{2}}((\lambda x)^{\frac{1}{2}}) dx$$

$$\begin{aligned}
&= \int_0^\infty \frac{1}{2} e^{-\frac{x+\lambda}{2}} \left(\frac{x}{\lambda}\right)^{\frac{q}{2}} I_q((\lambda x)^{\frac{1}{2}}) dx \\
&= \int_0^\infty \frac{1}{2} e^{-\frac{x+2\bar{u}}{2}} \left(\frac{x}{2\bar{u}}\right)^{\frac{q}{2}} I_q((2\bar{u}x)^{\frac{1}{2}}) dx \\
&= \int_0^\infty \frac{1}{2} e^{-\bar{u}-\bar{v}} \left(\frac{\bar{v}}{\bar{u}}\right)^{\frac{q}{2}} I_q(2(\bar{u}\bar{v})^{\frac{1}{2}}) dx \\
&= \int_0^\infty \frac{1}{2} e^{-\bar{u}-\bar{v}} \left(\frac{\bar{v}}{\bar{u}}\right)^{\frac{q}{2}} I_q(2(\bar{u}\bar{v})^{\frac{1}{2}}) 2dv \\
&= \int_0^\infty e^{-\bar{u}-\bar{v}} \left(\frac{\bar{v}}{\bar{u}}\right)^{\frac{q}{2}} I_q(2(\bar{u}\bar{v})^{\frac{1}{2}}) (c-\bar{B}) dr(s). \qquad (5.15)
\end{aligned}$$

As a result, considering (5.15) together with the fact that $A(s, s+\Delta)$, $B(s, s+\Delta)$, u, $\bar{u}$, and c are constant, the expectation (5.10) can be written as:

$$E(e^{r(s)\bar{B}-ln(\bar{A})}) =$$

$$
\begin{aligned}
&= \int_0^\infty \frac{c}{\bar{A}}\Big(\frac{c-\bar{B}}{c}\Big)^{-q} e^{\bar{u}-u} e^{-\bar{u}-\bar{v}}\Big(\frac{\bar{v}}{\bar{u}}\Big)^{\frac{q}{2}} I_q(2(\bar{u}\bar{v})^{\frac{1}{2}})dr(s) \\
&= \frac{c}{\bar{A}}\Big(\frac{c-\bar{B}}{c}\Big)^{-q} e^{\bar{u}-u}\frac{1}{c-\bar{B}}\int_0^\infty e^{-\bar{u}-\bar{v}}\Big(\frac{\bar{v}}{\bar{u}}\Big)^{\frac{q}{2}} I_q(2(\bar{u}\bar{v})^{\frac{1}{2}})(c-\bar{B})dr(s) \\
&= \frac{c}{\bar{A}}\Big(\frac{c-\bar{B}}{c}\Big)^{-q} e^{\bar{u}-u}\frac{1}{c-\bar{B}} \\
&= \frac{1}{\bar{A}}\Big(\frac{c}{c-\bar{B}}\Big)^{q+1} e^{\frac{u\bar{B}}{c-\bar{B}}} \\
&= \frac{1}{A(s,s+\Delta)}\Big(\frac{c}{c-B(s,s+\Delta)}\Big)^{q+1} e^{\frac{uB(s,s+\Delta)}{c-B(s,s+\Delta)}}. \qquad (5.16)
\end{aligned}
$$

Hence, the expression (5.6) is equivalent to:

$$
E(R(r(s),s,s+\Delta)^{TR}) \quad =
$$

$$
= \quad \frac{1}{A(s,s+\Delta)}\Big(\frac{c}{c-B(s,s+\Delta)}\Big)^{q+1} e^{\frac{uB(s,s+\Delta)}{c-B(s,s+\Delta)}} - 1. \qquad (5.17)
$$

Furthermore, expression (5.4) can be written as:

$$
\Big(\frac{\mathcal{E}^e_{s+\Delta}-\mathcal{E}^e_s}{\mathcal{E}^e_s}\Big)_2 \quad =
$$

$$
= \quad \frac{\frac{1}{\bar{A}}\left(\frac{c}{c-\bar{B}}\right)^{q+1} e^{\frac{u\bar{B}}{c-\bar{B}}} - \frac{1}{\bar{A}^*}\left(\frac{c^*}{c^*-\bar{B}^*}\right)^{q^*+1} e^{\frac{u^*\bar{B}^*}{c^*-\bar{B}^*}}}{\frac{1}{\bar{A}^*}\left(\frac{c^*}{c^*-\bar{B}^*}\right)^{q^*+1} e^{\frac{u^*\bar{B}^*}{c^*-\bar{B}^*}}}. \qquad (5.18)
$$

Calculations lead to an expected future one period total return in the long-run of:

$$
Yl = \lim_{s\to\infty}\frac{1}{A(s,s+\Delta)}\Big(\frac{c}{c-B(s,s+\Delta)}\Big)^{q+1} e^{\frac{uB(s,s+\Delta)}{c-B(s,s+\Delta)}} - 1 \quad =
$$

$$
= \quad \frac{1}{A(s,s+\Delta)}\Big(\frac{\frac{2\kappa}{\sigma^2}}{\frac{2\kappa}{\sigma^2}-B(s,s+\Delta)}\Big)^{\frac{2\kappa\theta}{\sigma^2}} - 1. \qquad (5.19)
$$

In the following, the current time t is chosen to be *zero* and the current exchange rate $\mathcal{E}_0$ is assumed to be equal to *one*. This standardization does not represent a restriction to the results presented in the latter part. Additionally, I chose the length of a period to be 0.01. I have to point out, that the choice of this particular value is absolutely arbitrary.

5.5 Characteristics of the ExpTR(Δ)

In this section, I present the results of the analysis of the ExpTR(0.01) based on the explicit form evolved in Section 5.4. That is, the characteristics of

$$E(R(r(s), s, s+0.01)^{TR}) \;=\; E(e^{r(s)B(s,s+0.01)-ln(A(s,s+0.01))})-1$$

using (5.17) are investigated.

As mentioned above, the expected future one period total return at time $s=0$ is already known from the term structure. Hence, the behavior of the expected yield at time $s=t=0$ is also known. Remember that Section 3.3 dealt with this topic. The reader may take that fact into consideration when studying the results and the figures below.

My investigations let me assume certain characteristics and a particular behavior of the ExpTR(0.01) when a change in one of the variables can be observed. Before presenting the results and visualizations of various examples, I argue why the results are valid. First of all, I introduced several intervals for possible and economic reasonable values of the variables. As a starting point, I refer to Chatterjee [4]. In this paper, quasi-maximum likelihood estimates of the model parameters are obtained by using a Kalman filter to calculate the likelihood function. Furthermore, estimates of σ^2 presented by Brown and Dybvig [3] were used to cut down the intervals to reasonable lengths. The bank base rates of the FED and the EZB of the last decades serve as a framework for the variables θ and r. Additionally, the conditions (3.54)-(3.59) on the variables from the paper of Cox, Ingersoll, and Ross [6] were used. A further condition stemming from the assumption that there exists a liquidity preference can be written as:

$$\begin{aligned}
R(r,t,\infty) &> \theta \Leftrightarrow \\
\frac{2\kappa\theta}{\gamma+\kappa+\lambda} &> \theta \Leftrightarrow \\
\frac{2\kappa}{\sqrt{(\kappa+\lambda)^2+2\sigma^2}+\kappa+\lambda} &> 1 \Leftrightarrow \\
2\kappa &> \sqrt{(\kappa+\lambda)^2+2\sigma^2}+\kappa+\lambda \Leftrightarrow \\
\kappa-\lambda &> \sqrt{(\kappa+\lambda)^2+2\sigma^2} \Leftrightarrow \\
\kappa^2-2\kappa\lambda+\lambda^2 &> \kappa^2+2\kappa\lambda+\lambda^2+2\sigma^2 \Leftrightarrow \\
\kappa\lambda &< -\frac{\sigma^2}{2} \Leftrightarrow \\
\sigma &< \sqrt{-2\kappa\lambda} \qquad (5.20)
\end{aligned}$$

Table 5.1 gives an overview of the intervals.

Variable	Interval	Increment
κ	[0.1,1]	0.05
θ	[0.005,0.08]	0.00025
r	[0.005,0.12]	0.00025
λ	[-(κ-0.05),-0.05]	0.05
σ	[0.05,min($\sqrt{-2\kappa\lambda}$,1)]	0.05

Table 5.1: Values of the variables of an economy

I calculated the partial derivatives analytically, however, it was not trivial to prove an unambiguous behavior for any composition of the variables. Instead, I used the partial derivatives to give evidence considering the characteristics of (5.17) by calculating the maximum and minimum, respectively, for any composition of the variables within the intervals mentioned above and proved the continuity of the partial derivatives. A positive minimum indicates that the expectation increases with an increase in the particular variable, while a negative maximum indicates that the expectation decreases with an increase in the particular variable. It was easy to show that all partial derivatives are continuous functions within the investigated intervals. This can be seen if one considers that any partial derivative of (5.17) is a combination of products, sums and fractions of the several elements of (5.17) and their partial derivatives. Considering the conditions on the variables, I split up equation (5.17) into several components and proved their continuity.[7] According to the characteristics of continuous functions, the continuity of the several components and their partial derivatives, respectively, and the examination, whether the several parts and their various combinations are well defined, are sufficient for the proof of continuity. Consequently, I could easily show the continuity of function (5.17).

Although my thesis lacks an analytical proof of the characteristics, the investigation of the partial derivatives for a number of compositions and the fact that

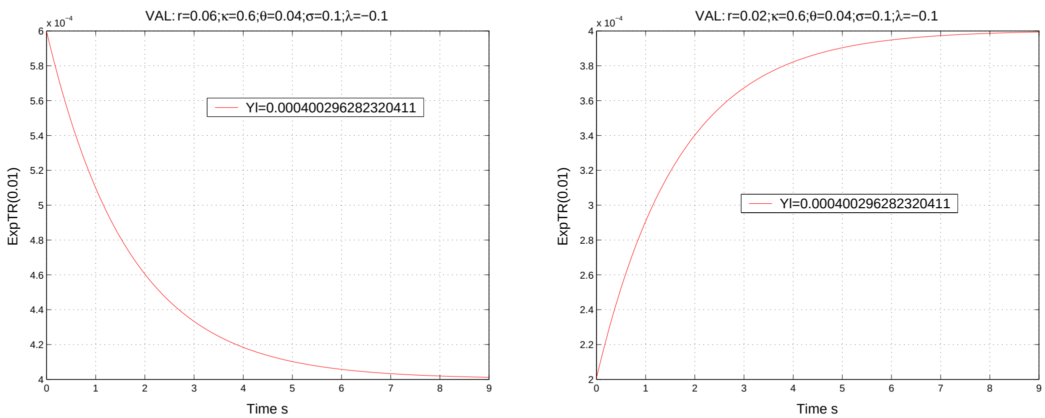
[7]For further details see Appendix B.6.

these partial derivatives are continuous functions do substantiate my propositions.

At this point, I present the behavior of the expectation.[8]

As mentioned in Section 5.2, I assume that the inequation (5.2) holds and that the expected future one period total return can be approximated by $e^{E(r(s)|r(t))0.01} - 1$. Considering (3.42), for $s \to 0$ the ExpTR(0.01) can be approximated by $e^{r \cdot 0.01} - 1$ (for $s \to \infty$ it can be approximated by $e^{\theta \cdot 0.01} - 1$). Furthermore, the approximation $e^{r \cdot 0.01} \approx 1 + r \cdot 0.01$ for small $r \cdot 0.01$ is used. According to Table 5.1, it can be assumed that the size of r is sufficiently small to allow for this approximation.

While the ExpTR(0.01) are rising if the $r \times 0.01$ is below the value in the long-run, the ExpTR(0.01) are falling if $r \times 0.01$ is in excess of (5.19).



(a) $r \cdot 0.01 >= Yl$

(b) $r \cdot 0.01 < Yl$

Figure 5.2: Structure dependent on current spot rate

[8] For MATLAB source file 'exptr.m' see Appendix C.

Moreover, several other comparative statics for the yield curve are obtained. Calculations have shown that an increase in the current interest rate increases the ExpTR(0.01) at any future time s. This can be easily interpreted if one considers that a bond's yield is a composition of the spot rate and a premium. A higher spot rate influences the expectations concerning the one period total returns, as a higher value of spot rate indicates greater yields. The long-term value of the spot rate, θ, has not changed and, therefore, the expected one period yields in the long-run have not changed very much. Hence, the effect is greater for the expectations in the relatively near future. The influence on the total return of a one period bond at time $s = 0$, which is determined by the term structure, is the same. Hence, the behavior of the expectation is consistent with the behavior of the term structure investigated in Section 3.3.

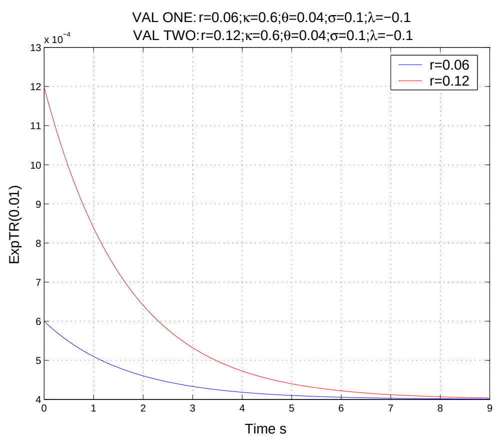


Figure 5.3: Effect of an increase in r on the ExpTR(0.01)

Similarly, an increase in the steady state mean θ increases the ExpTR(0.01), but here the effect is greater for the expectations in the relatively distant future as the long-term value θ has changed, whereas the current interest rate r has not. The influence on the total return of a one period bond at time $s = 0$ is the same. Hence, the behavior of the expectation is consistent with the behavior of the term structure investigated in Section 3.3.

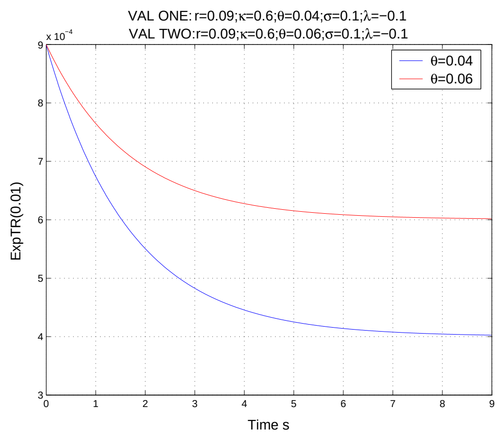
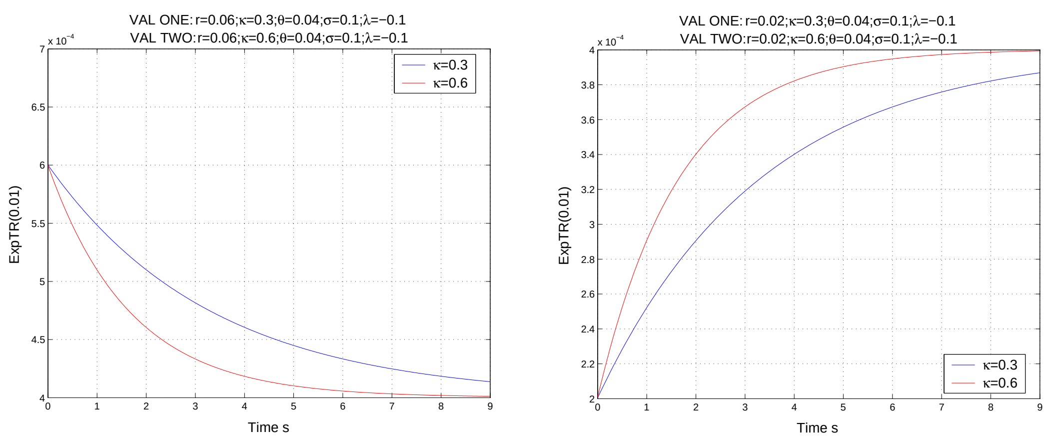


Figure 5.4: Effect of an increase in θ on the ExpTR(0.01)

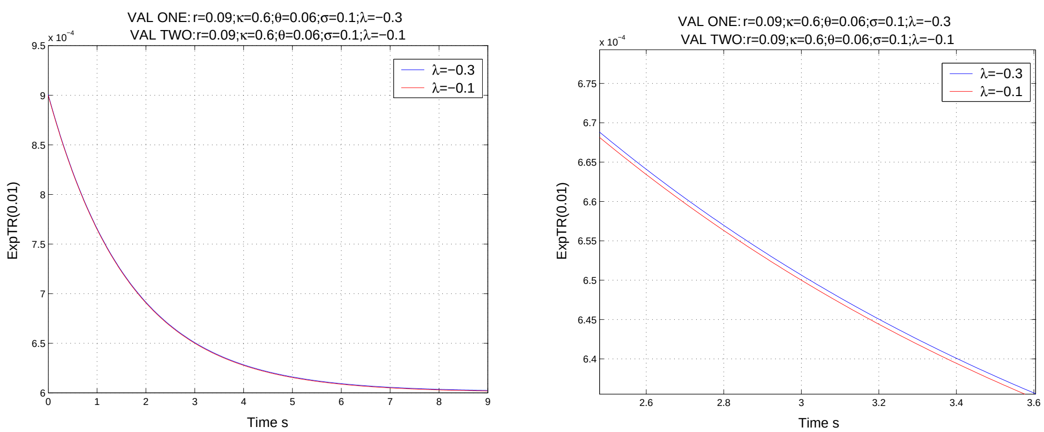
The effect of a change in κ may be of either sign depending on the current interest rate, that is, the expected value is an increasing function of the speed of adjustment parameter κ if the spot rate is less than θ and a decreasing function of κ if the spot rate is greater than θ respectively. This can be seen if one considers that a higher value of κ means that the spot rate adjusts faster to a higher/lower level. The influence on the total return of a one period bond at time $s = 0$ is the same. Hence, the behavior of the expectation is consistent with the behavior of the term structure investigated in Section 3.3.



(a) $r >= \theta$ (b) $r < \theta$

Figure 5.5: Effect of an increase in κ on the ExpTR(0.01)

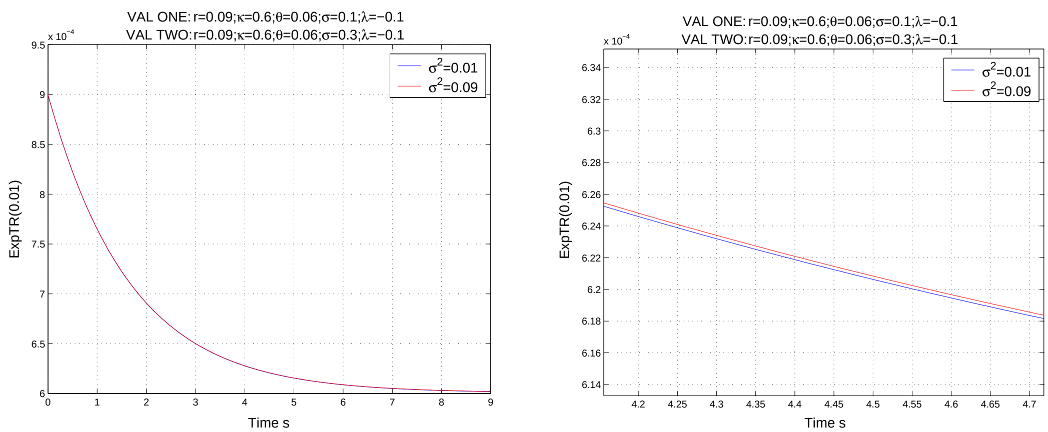
The ExpTR(0.01) decrease as λ increases. This can be easily seen as one remembers that higher values of λ indicate lower premiums as λ is the market value of risk. As λ increases (or $|\lambda|$ decreases), the value of risk decreases. This development influences the expectation of the agent. A lower market value of risk decreases the expected one period yield and, consequently, the total return. The influence on the total return of a one period bond at time $s = 0$ is the same. Hence, the behavior of the expectation is consistent with the behavior of the term structure investigated in Section 3.3.



(a) (b) Zoom

Figure 5.6: Effect of an increase in λ on the ExpTR(0.01)

The effect of an increase in σ^2, however, leads to a surprising result: it increases the ExpTR(0.01). The effect of a change of σ^2 on the (expected) yield at time $s = 0$ is already known from Section 3.3. A higher value of the variance of the interest rate, σ^2, indicates more uncertainty about future real production opportunities, and thus more uncertainty about future consumption. As a consequence, the guaranteed claim in a bond is valued more highly by investors and the yield decreases. Compared with that, the ExpTR(0.01) increases at any time s.



(a) (b) Zoom

Figure 5.7: Effect of an increase in σ^2 on the ExpTR(0.01)

Note that another choice of the length of one period, e.g. $\Delta = 0.1$, influences the value of the ExpTR(Δ) in the long-run. This follows from (5.19). The value increases as the term increases. Moreover, the expected future one period total return at any time increases as the term increases. The result is not surprising, as bonds with a longer maturities achieve a higher total return.

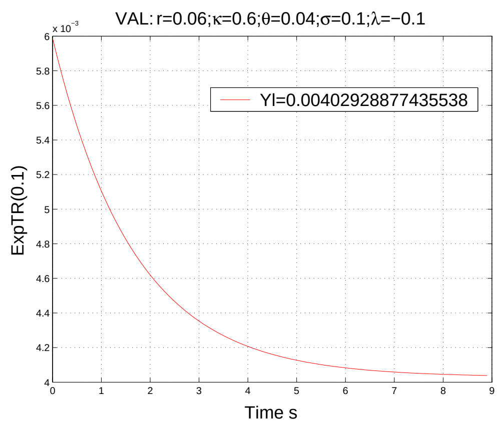


Figure 5.8: Effect of an increase in the length of one period, e.g. $\Delta = 0.1$

Moreover, a change of the length of one period may change the results stated above, although various examples do not indicate that for changes of the variables r, θ, λ, and κ. However, longer periods change the influence of changes in σ^2 on the expectations. For example, with $\Delta = 2.5$, and given the particular values of the variables as in Figure 5.7, an increase in σ^2 leads to a contrary statement. Here, an increase in the variance of the spot rate leads to a decrease of the ExpTR(2.5).

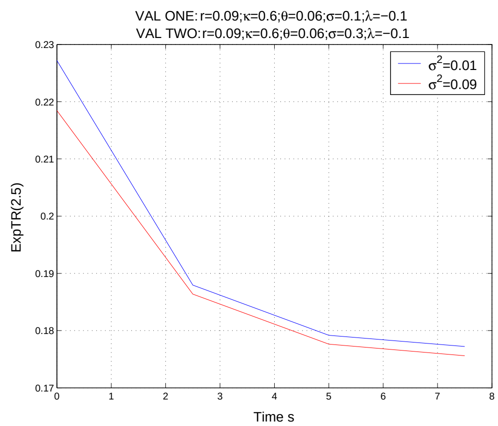


Figure 5.9: Effect of an increase in σ^2 on the ExpTR(2.5)

Basically, that phenomenon raises the following questions:
(1) Why does an increase in the variance of the spot rate, which usually indicates a higher uncertainty about real production opportunities, result in *higher* expected one period yields in the future if the length of the period is chosen to be 0.01?
(2) Why does the length of the period seem to influence this result in the one or the other direction?

Supplementary investigations may lead to further and deeper insights, however, I hold the view that it is not trivial to find a reasonable interpretation of that behavior and would be beyond the scope of that thesis. I did not undertake supplementary investigations, because I focused more attention on the interpretation of the expected exchange rates and the particular expected depreciation rates as exemplified in Chapter 7. Without a reasonable explanation, however, one can only state that this surprising result may imply certain, so far unknown, problems within this approach.

Chapter 6

The Influence of the Expectations on the Spot Exchange Rate

There are several approaches which describe the mechanism of changes of both the spot exchange rate and the future expectations. One possible way to argue would be to assume that because of the expected development of the exchange rate, e.g. a depreciation of the currency, the agents anticipate the development by speculative purchases of the foreign currency. Hence, the spot exchange rate is influenced by the expectations. On the contrary, one could also argue that because of risk-averse agents the expected development is not anticipated by the agents.

This chapter is dedicated to the question of how changes of the factors of the economies influence the expectations and the spot exchange rate. However, I confine myself to the following situation: I analyze the behavior of the spot exchange rate and the expected exchange rates when a particular value of the current interest rate at a future point is observed. The other variables are assumed to remain unchanged.

It is well-known that changes of the expected exchange rate lead to changes of the spot exchange rate. Most known models, however, only take one expected exchange rate into consideration when calculating the effect on the spot exchange rate. In my approach the whole path will influence the spot exchange rate. I assume that the expectations are more rigid than the spot exchange rate, that is, the expectations are assumed to change less than the spot exchange rate when a certain value of the spot rate appears. Moreover, it is assumed that the current interest rate affects on the spot exchange rate in a particular way, such that the resulting new expectations of the exchange rate due to the term structure at that future time s do differ as little as possible from the expectations formed initially at time t. Figure 6.1 displays the idea of choosing a particular spot exchange rate such that the differences between the expected exchange rates are minimized.

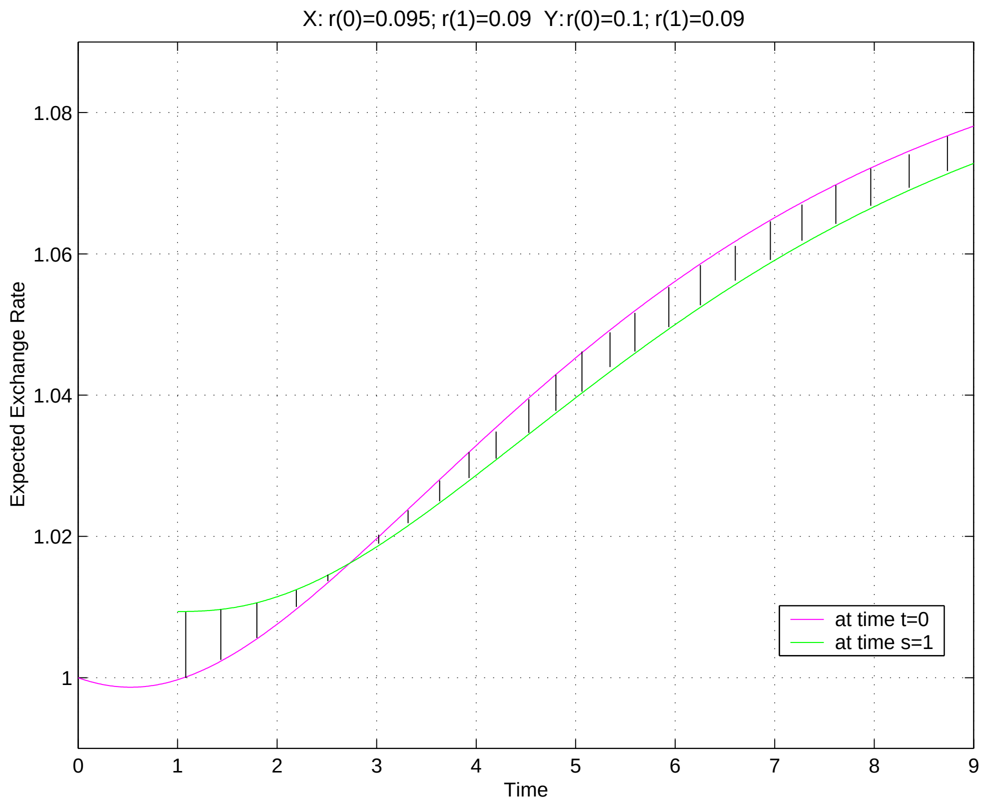


Figure 6.1: The spot exchange rate in the future

In order to ensure that these differences do not balance themselves out, the differences are squared. Furthermore, I assume that expectations in the long-run are less rigid and adjust more easily. Hence, the expectations are weighted differently by introducing the weight $\frac{1}{T}$. As can be seen in the figure above, the spot rates were initially 9.5% and 10% respectively. In country X, a decrease of the spot rate at time $s = 1$ can be observed while in country Y an increase is seen. The spot exchange rate is supposed to adjust to a particular level, such that the marked area (in accordance with the weights) is minimized. As mentioned above, the other variables describing both economies are assumed to remain unchanged. This implies, however, that the results presented below are only reasonable if the investigated point of time is in the near future. This stems from the assumption that changes in the other variables only appear in the long-run.

In technical terms these assumptions can be expressed as follows:

$$\mathcal{E}_s = \arg\min\left\{\int_s^\infty \left[(\mathcal{E}_T^e|\mathcal{E}_t) - (\mathcal{E}_T^e|\mathcal{E}_s)\right]^2 \frac{1}{T} dT\right\} \tag{6.1}$$

$$= \arg\min\left\{\int_s^\infty \left[\frac{P^*(r^*(t),t,T)}{P(r(t),t,T)}\mathcal{E}_t - \frac{P^*(r^*(s),s,T)}{P(r(s),s,T)}\mathcal{E}_s\right]^2 \frac{1}{T} dT\right\}, \tag{6.2}$$

where $(\mathcal{E}_T^e|\mathcal{E}_s)$ stands for the expected exchange rate at time T given the information at time s, that is the current interest rate and the spot exchange rate $\mathcal{E}_s$.

The definite integral

$$F(y) = \int_a^b f(x,y)dx \tag{6.3}$$

is called *integral with parameter*. If the function is defined within an interval $[c,e]$ and the integrand is continuous within $[a,b] \times [c,e]$ and is partially differentiable with respect to y, then

$$\frac{d}{dy}\int_a^b f(x,y)dx = \int_a^b \frac{\partial f(x,y)}{\partial y} dx \tag{6.4}$$

for an arbitrary y within the interval $[c,e]$.[1] A minimum of (6.3) needs to satisfy the following conditions:

$$F'(y) = \int_a^b \frac{\partial f(x,y)}{\partial y} dx = 0 \tag{6.5}$$

$$F''(y) = \int_a^b \frac{\partial^2 f(x,y)}{\partial y^2} dx > 0. \tag{6.6}$$

The initial type of problem can be interpreted as finding the argument minimizing the integral on the right hand side of (6.2), which depends on the parameter $\mathcal{E}_s$. Although the integral in (6.2) is an improper integral as the upper limit is infinite, I restricted my investigation of the solution of that integral to a closed interval $[t,R]$. I hold the view that this restriction does not alter the usefulness of that approach, because expectations loose their meaningfulness as the point of time is in the relatively distant future. Moreover, the weight decreases the importance of those expectations for the determination of $\mathcal{E}_s$. Finally, the restriction of the investigation to a close interval is reasonable for computational purposes. If one considers that the integrand of (6.2) is a continuous function within the investigated interval and can be partially differentiated with respect to $\mathcal{E}_s$, the

[1] For further information see [16].

sufficient conditions (6.5) and (6.6) can be used. The conditions can be written as follows:

$$\int_s^R \frac{-2}{T}\Big[\frac{P^*(r^*(t),t,T)}{P(r(t),t,T)}\mathcal{E}_t - \frac{P^*(r^*(s),s,T)}{P(r(s),s,T)}\mathcal{E}_s\Big]\frac{P^*(r^*(s),s,T)}{P(r(s),s,T)}dT = 0 \quad (6.7)$$

$$\int_s^R \frac{2}{T}\frac{P^*(r^*(s),s,T)^2}{P(r(s),s,T)^2}dT > 0. \quad (6.8)$$

It can be easily seen that (6.8) is satisfied. Consequently, the spot exchange rate at time s can be calculated by:

$$\mathcal{E}_s = \frac{\int_s^R \frac{P^*(r^*(t),t,T)}{P(r(t),t,T)}\frac{P^*(r^*(s),s,T)}{P(r(s),s,T)}\frac{\mathcal{E}_t}{T}dT}{\int_s^R \frac{P^*(r^*(s),s,T)^2}{P(r(s),s,T)^2}\frac{1}{T}dT}. \quad (6.9)$$

Finding an analytical solution of the integrals presented in (6.9) is not a trivial problem. Using the strategy of adaptive quadrature, however, the value can be calculated easily.[2]

If, instead of an actual spot rate at time s the expectation of the spot rate at time s given the spot rate at the current time t (equation (3.42)) is used, the value of (6.9) can be interpreted as another expectation of the exchange rate at time s. It follows:

$$\mathcal{E}_s^e = \frac{\int_s^R \frac{P^*(r^*(t),t,T)}{P(r(t),t,T)}\frac{P^*(E(r^*(s)|r^*(t)),s,T)}{P(E(r(s)|r(t)),s,T)}\frac{\mathcal{E}_t}{T}dT}{\int_s^R \frac{P^*(E(r^*(s)|r^*(t)),s,T)^2}{P(E(r(s)|r(t)),s,T)^2}\frac{1}{T}dT}. \quad (6.10)$$

Investigations of these expectations were not done. However, the reader may take into consideration that this approach would be another alternative to the expectations formed in Chapter 4 and 5.

[2]For MATLAB source file 'nextperiod.m' see Appendix C.

Chapter 7

Examples: Analysis and Visualization

In this chapter, the results acquired in Chapter 4 and Chapter 5 are used to interpret the term structure, the expected exchange rate, and the two elaborated expected depreciation rates of various examples of two economies with particular values for the several variables.[1] Furthermore, I present two examples visualizing the results of Chapter 6. Each choice of the values of the variables takes into consideration the various conditions stemming from the model of the economy. At this point, I list the conditions mentioned in the chapters above again:

$$\begin{aligned}
\kappa\theta &\geq 0 \\
\sigma^2 &> 0 \\
\lambda &< 0 \\
\kappa &> 0 \\
\theta &> 0 \\
\kappa &> |\lambda| \\
\sigma &< \sqrt{-2\kappa\lambda}
\end{aligned}$$

Furthermore, it is important to know that the examples are chosen such that the long-term yields are the same at the most part. This makes at easier to investigate the influence of single variables on the expectations. The term structure, the expected exchange rates and the expected depreciation rates are visualized.

[1]For MATLAB source file 'exchangerates.m' see Appendix C.

7.1 Term Structure, Expected Exchange and Depreciation Rates

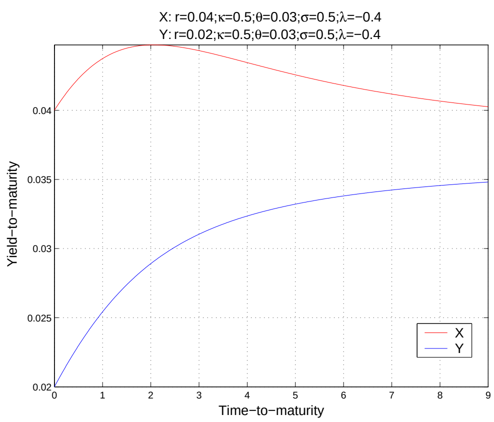
In this section, I present various examples of economies including their particular term structure, the expected exchange rates, and the expected depreciation rates.

Example 1:

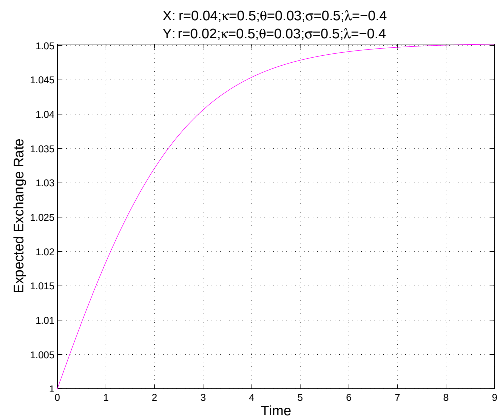
Variable	Country X	Country Y
r:	0.04 (4%)	0.02 (2%)
κ:	0.5	0.5
θ:	0.03 (3%)	0.03 (3%)
σ:	0.5	0.5
λ:	-0.4	-0.4
$\frac{2\kappa\theta}{\gamma+\kappa+\lambda}$:	0.0368 (3.68%)	0.0368 (3.68%)
$\frac{\kappa\theta}{\kappa+\lambda}$:	0.15 (15%)	0.15 (15%)

Table 7.1: Example 1

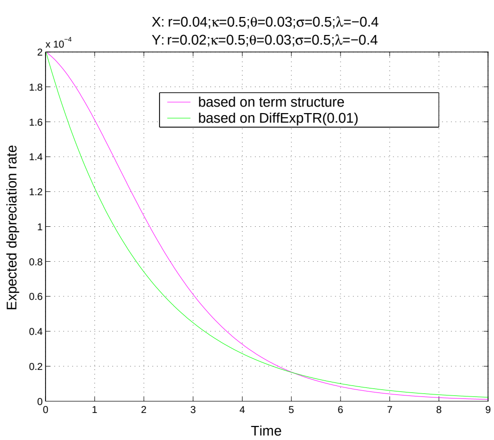
The economies of both countries are very similar. They only differ from each other in the difference of the current interest rate. Consequently, the long-term yields (3.52) and the critical values (3.53) are the same respectively. Figure 7.1 (a) shows the term structure in both countries. As one can see, the spot rate difference leads to higher yields in country X than in country Y. Moreover, while the yield curve of country X is humped as the spot rate is between the values (3.52) and (3.53), the yield curve of country Y is increasing as the spot rate is below the long-term yield. According to this, it is not surprising that the representative agents expect a depreciation of currency X. These expectations can be seen in Figure 7.1 (b). Because of the fact that the long-term yields are the same, one would expect the exchange rate to appreciate in the long-run and to find its level of the current exchange rate. This can not be seen in the figures, as the investigated horizon is not long enough. In figure 7.1 (c), however, it can be seen that the expected depreciation rate decreases as the absolute change of the expected exchange rate from one period to another decreases. Moreover, the expected depreciation rate seems to become zero. The purple line describes that process. The expected depreciation rate described by the green line evolves from the expected future one period total return differentials. The spot rate difference leads to positive DiffExpTR(0.01). In the long-run, this expected depreciation rate also becomes zero. This development can be easily explained if one considers that the long-term values of the spot rates are the same, namely 3%. As mentioned in Section 5.5, the expectation in the long-run can be approximated by $e^{\theta \cdot 0.01} - 1$. Consequently, the lack of a difference in the expected future one period total return in the long-run leads to a depreciation rate of almost zero. Both expected depreciation rates show a similar structure and the differences are quite minimal. Finally, the different development of the two expected depreciation rates is inconsistent with the expected equality stated by the expectations hypothesis.



(a) Term Structure



(b) Expected Exchange Rate



(c) Expected Depreciation Rate

Figure 7.1: Example 1

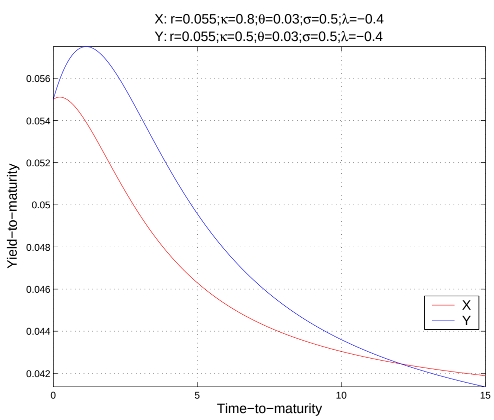
Example 2:

Variable	Country X	Country Y
r:	0.055 (5.5%)	0.055 (5.5%)
κ:	0.8	0.5
θ:	0.03 (3%)	0.03 (3%)
σ:	0.5	0.5
λ:	-0.4	-0.4
$\frac{2\kappa\theta}{\gamma+\kappa+\lambda}$:	0.0396 (3.96%)	0.0368 (3.68%)
$\frac{\kappa\theta}{\kappa+\lambda}$:	0.06 (6%)	0.15 (15%)

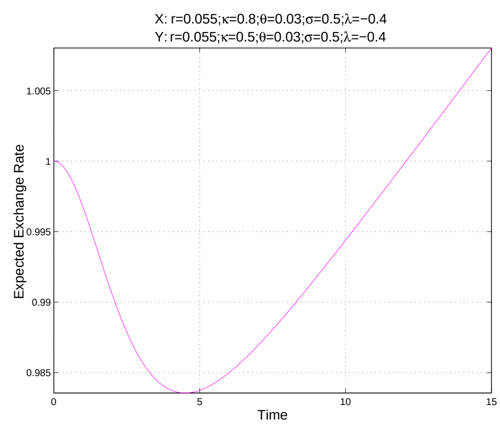
Table 7.2: Example 2

The economies of both countries are very similar. They only differ from each other in the difference of the parameters describing the respective speed of adjustment of the spot rate. A difference in κ, however, indicates different values of the long-term yields (3.52) and the critical values (3.53). Figure 7.2 (a) shows the term structure in both countries. As one can see, although the spot rates are the same, the difference in κ leads to different critical values. Apparently, the spot rate of country X is in excess of the long-term yield and below the critical value. Consequently, the term structure is humped. The term structure of country Y is also humped, however, the maximum is lower than the maximum of country X's term structure. This difference accounts for the appreciation of the expected exchange rate as shown in Figure 7.2 (b). A higher speed of adjustment in country X explains the positive value of the expected rate of depreciation at approximately time 4.5 (see Figure 7.2(c)). Nevertheless, the expected exchange rate is still lower than the spot exchange rate at time t, as the yields of country X are still higher than those of country Y. In the long-run, however, the higher value of κ leads to an intersection of both of the yield curves. At this point, the expected rate of depreciation also leads to an absolute depreciation of the expected exchange rate. In the distant future, however, one would expect to observe an appreciation of the expected exchange rate, as the long-term yield of country X is higher than the long-term yield of country Y. This can not be seen in the figure as the investigated horizon is not long enough. The green line describes a different development of the expected rate of depreciation. A depreciation of the exchange rate is not expected. This can be seen if one considers that the speed of adjustment in country X is higher, the long-term values of the spot rates are the same and the spot rates are in excess of the particular θ. It is not surprising that the rate seems to become zero in the long-run as the expectation in the long-run can be approximated by $e^{\theta \cdot 0.01} - 1$. Both expected depreciation rates show a similar structure, however, the expected depreciation rate based on the term structure seems to be more volatile. Finally, the different development of the two expected depreciation rates is inconsistent with the expected equality stated by the expectations hypothesis.

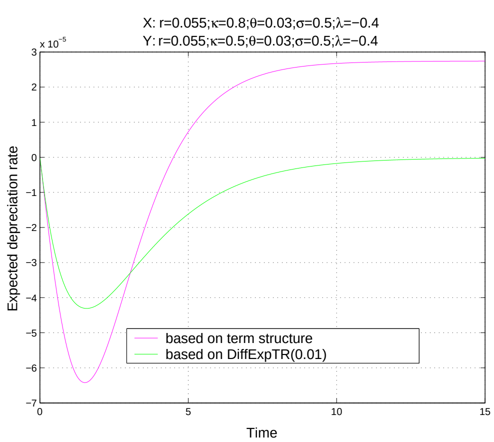
The phenomenon of an appreciation in the beginning, followed by a depreciation is not unknown at all. One can often observe the exchange rate to overshoot and undershoot its final level respectively. One explanation is that the equilibrium on the foreign exchange rate market is reached faster than one on the market for goods. For further information see Krugman and Obstfeld [11].



(a) Term Structure



(b) Expected Exchange Rate



(c) Expected Depreciation Rate

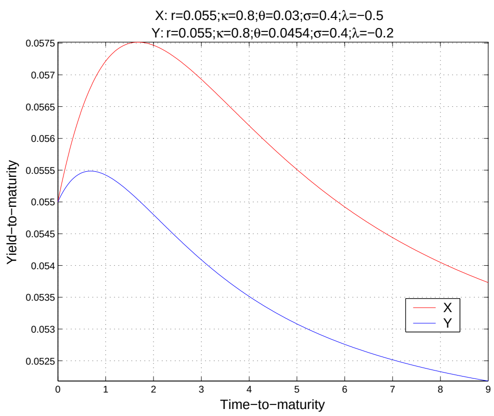
Figure 7.2: Example 2

Example 3:

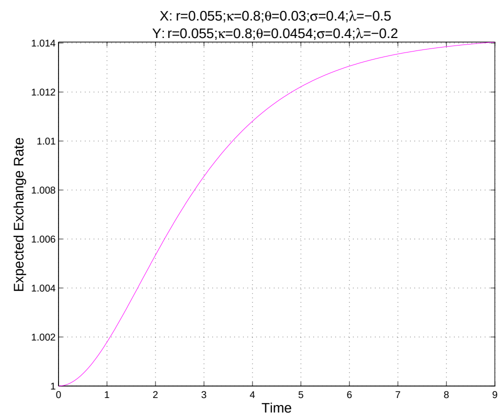
Variable	Country X	Country Y
r:	0.055 (5.5%)	0.055 (5.5%)
κ:	0.8	0.8
θ:	0.03 (3%)	0.0454 (4.54%)
σ:	0.4	0.4
λ:	-0.5	-0.2
$\frac{2\kappa\theta}{\gamma+\kappa+\lambda}$:	0.0510 (5.1%)	0.0510 (5.1%)
$\frac{\kappa\theta}{\kappa+\lambda}$:	0.08 (8%)	0.0605 (6.05%)

Table 7.3: Example 3

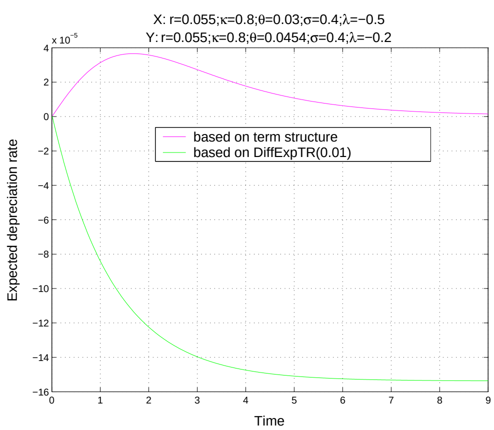
The long-term values of the spot rates differ from each other as well as the market values of risk. The risk of the market in country X is valued higher than that in country Y. This example, however, represents a situation where the long-term yields (3.52) are the same. Basically, in the long-run the expected exchange rate is supposed to be the same as the spot exchange rate at time $t = 0$. Figure 7.3 (a) shows the term structure in both countries. As one can see, the lower value of $|\lambda^*|$ of country Y leads to lower yields, although θ^* is higher. The absolute effect of the difference in λ seems to be stronger than the effect of the difference in θ, as the higher value of θ^* would indicate higher yields in country Y compared to the yields paid in country X. The term structure of both countries is humped. The higher yields in country X are the reason why the exchange rate is expected to depreciate. Figure 7.3 (b) shows this expectation. As a result, the expected rate of depreciation is positive as the purple line in Figure 7.3 (c) verifies. The fact that κ and κ^*, the respective speed of adjustment, and the long-term yields are the same in both economies can explain why the expected rate of depreciation becomes almost zero. Because of the fact that the long-term yields are the same, one would expect the exchange rate to appreciate in the long-run and to find its level of the today's exchange rate. This can not be observed as the time horizon is not long enough. As 7.3 (c) shows, the expected rate of depreciation based on the expected future one period total returns is negative. That speaks for the assumption that the absolute effect of a higher value of θ^* in country Y is higher than the absolute effect of a higher value of λ^* on the ExpTR(0.01)*. The expected rate of depreciation seems to converge to a value which can be approximated by $e^{(\theta-\theta^*)\cdot 0.01} - 1$. This can be easily interpreted if one considers that the expectation in the long-run can be approximated by $e^{\theta \cdot 0.01} - 1$. This example, however, shows that the expected depreciation rates calculated using the results of Chapter 4 and Chapter 5 develop totally differently. Finally, the different development of the two expected depreciation rates is inconsistent with the expected equality stated by the expectations hypothesis.



(a) Term Structure



(b) Expected Exchange Rate



(c) Expected Depreciation Rate

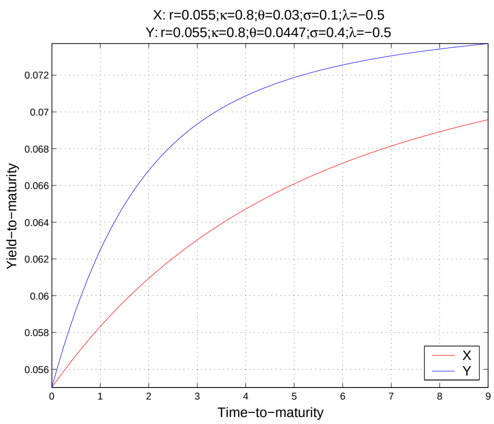
Figure 7.3: Example 3

Example 4:

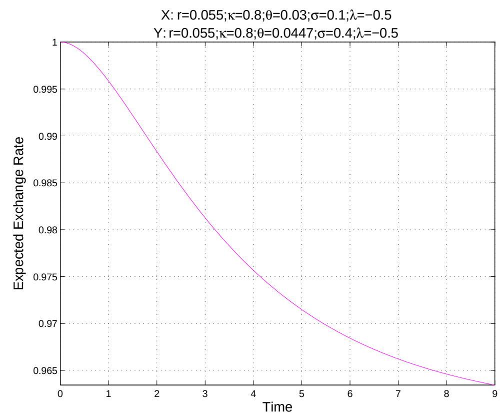
Variable	Country X	Country Y
r:	0.055 (5.5%)	0.055 (5.5%)
κ:	0.8	0.8
θ:	0.03 (3%)	0.0447 (4.47%)
σ:	0.1	0.4
λ:	-0.5	-0.5
$\frac{2\kappa\theta}{\gamma+\kappa+\lambda}$:	0.0760 (7.6%)	0.0761 (7.61%)
$\frac{\kappa\theta}{\kappa+\lambda}$:	0.08 (8%)	0.1192 (11.92%)

Table 7.4: Example 4

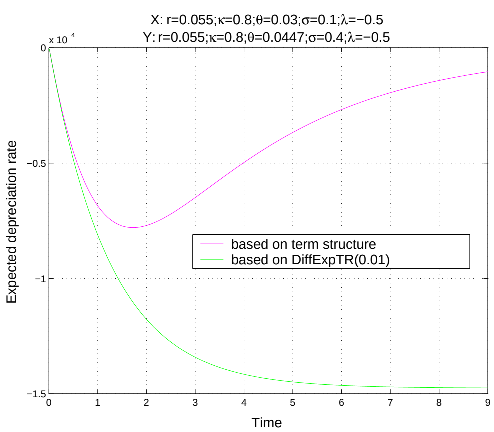
The long-term values of the spot rates differ from each other as well as the variances of the spot rates. The variance of the spot rate in country Y is higher than that in country X. That is, the guaranteed claim in a bond is valued more highly by investors in country Y. This example, however, represents a situation where the long-term yields (3.52) are the same. Basically, in the long-run the expected exchange rate is supposed to be the same as the spot exchange rate at time $t = 0$. Figure 7.4 (a) shows the term structure in both countries. As one can see, the higher value of θ^* of country Y leads to higher yields, although $(\sigma^*)^2$ is higher. The absolute effect of the difference in θ seems to be stronger than the effect of the difference in σ^2, as the higher value of $(\sigma^*)^2$ would indicate lower yields compared to those paid in country X. The term structure of both countries is increasing. The higher yields in country Y are the reason why the exchange rate is expected to appreciate. Figure 7.4 (b) shows this expectation. As a result, the expected rate of depreciation is negative as the purple line in Figure 7.4 (c) verifies. The fact that κ and κ^*, the respective speed of adjustment, and the long-term yields are the same in both economies can explain why the expected rate of appreciation seems to tend to become almost zero. Because of the fact that the long-term values are the same, one would expect the exchange rate to depreciate in the long-run and to find its level of the today's exchange rate. This can not be observed as the time horizon is not long enough. As 7.4 (c) shows, the expected rate of depreciation based on the DiffExpTR(0.01) is negative. That can be easily seen if one remembers that a higher value of θ^* in country Y leads to higher expectations as well as the surprising result that with $\Delta = 0.01$ a higher value of $(\sigma^*)^2$ also leads to higher expectations. The expected rate of depreciation seems to converge to a value which can be approximated by $e^{(\theta-\theta^*)\cdot 0.01} - 1$. This can be easily interpreted if one considers that the expectation in the long-run can be approximated by $e^{\theta \cdot 0.01} - 1$. Finally, the different development of the two expected depreciation rates is inconsistent with the expected equality stated by the expectations hypothesis.



(a) Term Structure



(b) Expected Exchange Rate



(c) Expected Depreciation Rate

Figure 7.4: Example 4

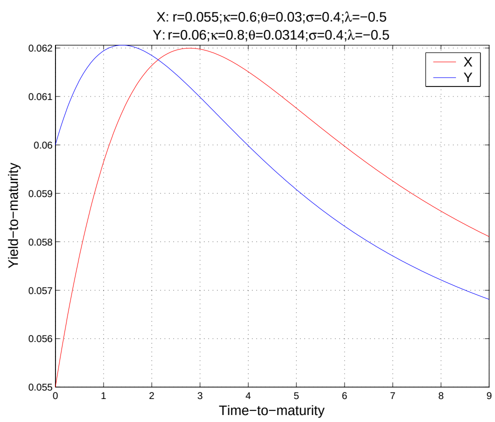
Example 5:

Variable	Country X	Country Y
r:	0.055 (5.5%)	0.06 (6%)
κ:	0.6	0.8
θ:	0.03 (3%)	0.0314 (3.14%)
σ:	0.4	0.4
λ:	-0.5	-0.5
$\frac{2\kappa\theta}{\gamma+\kappa+\lambda}$:	0.0534 (5.34%)	0.0534 (5.34%)
$\frac{\kappa\theta}{\kappa+\lambda}$:	0.18 (18%)	0.0837 (8.37%)

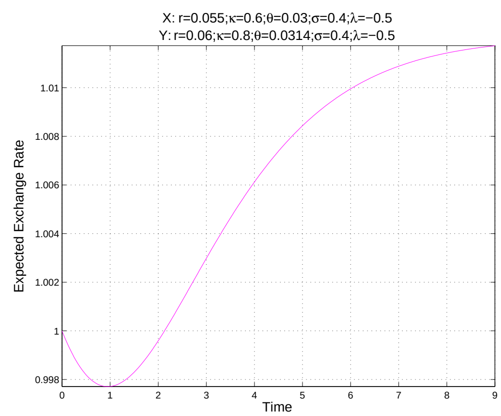
Table 7.5: Example 5

The spot rates differ from each other as well as the parameters describing the speed of adjustment of the spot rates and the long-term values θ and θ^*. The speed of adjustment of the spot rate in country Y is higher than that in country X. This example, however, represents a situation where the long-term yields (3.52) are the same. Basically, in the long-run the expected exchange rate is supposed to be the same as the spot exchange rate at time $t = 0$. Figure 7.5 (a) shows the term structure in both countries. The term structure of both countries is humped. As one can see, the higher value of the spot rate of country Y leads to higher yields in the short-run. The higher yields in country Y are the reason why the exchange rate is expected to appreciate. The fact that the long-term yields are the same in both economies and that the speed of adjustment in country Y is higher can explain why the expected exchange rate depreciates as the yields with longer time-to-maturity are higher in country X. Figure 7.5 (b) shows this expectation. As a result, the expected rate of depreciation which is negative in the short-run, becomes positive, and seems to become almost zero in the long-run as the purple line in Figure 7.5 (c) verifies. As 7.5 (c) shows, the expected rates of depreciation (based on the expected future one period total returns) show a similar development. The lower spot rate and the lower value of θ outweigh the contrary effect of a lower κ (as $r > \theta$) in country X. A lower κ would increase the ExpTR(0.01), while a lower value of θ would decrease the ExpTR(0.01) compared to ExpTR(0.01)*. The effect of the change of θ seems to be stronger. For only a short period of time the differentials of the expected future one period total returns are positive. In the long-run the expected rate of depreciation seems to converge to a value which can be approximated by $e^{(\theta-\theta^*)\cdot 0.01} - 1$. This can be easily interpreted if one considers that the expectation in the long-run can be approximated by $e^{\theta \cdot 0.01} - 1$. The expected depreciation rates show a similar structure, however, the absolute changes of the expected exchange rates differ from each other. Finally, the different development of the two expected depreciation rates is inconsistent with the expected equality stated by the expectations hypothesis.

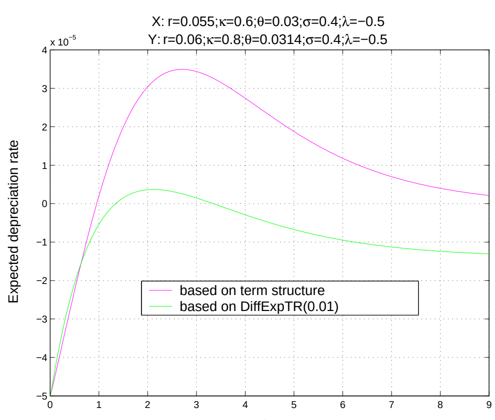
The phenomenon of an appreciation in the beginning, followed by a depreciation is not unknown at all. One can often observe the exchange rate to overshoot and undershoot its final level respectively. One explanation is that the equilibrium on the foreign exchange rate market is reached faster than one on the market for goods. For further information see Krugman and Obstfeld [11].



(a) Term Structure



(b) Expected Exchange Rate



(c) Expected Depreciation Rate

Figure 7.5: Example 5

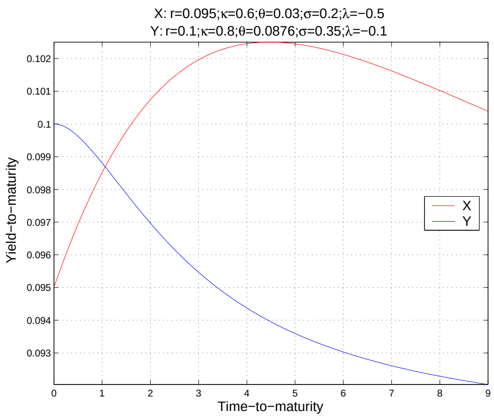
Example 6:

Variable	Country X	Country Y
r:	0.095 (9.5%)	0.10 (10%)
κ:	0.6	0.8
θ:	0.03 (3%)	0.0876 (8.76%)
σ:	0.2	0.35
λ:	-0.5	-0.1
$\frac{2\kappa\theta}{\gamma+\kappa+\lambda}$:	0.09 (9%)	0.09 (9%)
$\frac{\kappa\theta}{\kappa+\lambda}$:	0.18 (18%)	0.1001 (10.01%)

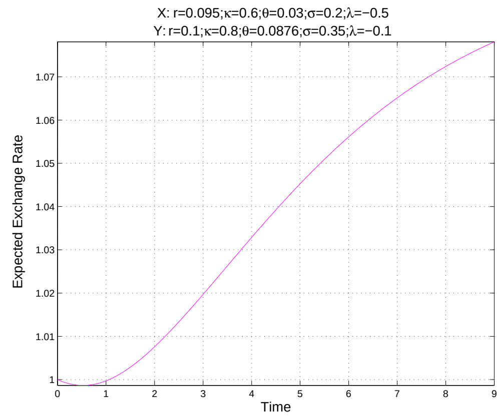
Table 7.6: Example 6

Every variable of country X is different from those of country Y. The expected long-term values of the spot rates differ from each other as well as the variances of the spot rates. Moreover, the spot rates, market values of risk and the parameters describing the respective speed of adjustment are different. However, the long-term yields are the same. Consequently, in the long-run the expected exchange rate is supposed to be the same as the spot exchange rate at time $t = 0$. The spot rate of country Y is higher than the spot rate of country X as well as the long-term value. The variance of the spot rate in country Y is higher than that in country X. That is, the guaranteed claim in a bond is valued more highly by investors in country Y. The market value of risk is higher in country X, where $|\lambda|$ is higher. The speed of adjustment is also higher in country Y. Figure 7.6 (a) shows the term structure in both countries. Both yield curves are humped. As one can see, the higher value of the spot rate in country Y leads to an appreciation of the expected exchange rate. After approximately 10 periods, however, the yields in country X are higher. Although a higher value of θ^* in country Y would lead to higher yields compared to those paid in country X, the effect of a higher variance of the spot rate, a lower market value of risk and a higher value of the speed of adjustment outweigh the effect of θ^*. This leads to yields which are lower than those of country X. Consequently, one expects the exchange rate to depreciate. Figure 7.6 (b) shows this expectation. As a result, the expected rate of depreciation is negative in the short-run and becomes positive in the long-run as the purple line in Figure 7.6 (c) verifies. The fact that the long-term yields are the same in both economies can explain why the expected rate of appreciation seems to tend to become almost zero. As mentioned above, the same values for the long-term yield would lead to a negative expected rate of depreciation (an appreciation) such that the expected exchange rate finds its level in the current exchange rate again. This can not be observed as the time horizon is not long enough. As 7.6 (c) shows, the expected rate of depreciation based on the DiffExpTR(0.01) is negative all the time. This can be easily seen if one remembers that a higher value of θ^* in country Y leads to higher expectations as well as the surprising result that a higher value of $(\sigma^*)^2$ also leads to higher expectations. A lower value of $|\lambda^*|$ would lead to lower expectations as well as a higher speed of adjustment in country Y with the spot rate higher than θ^*. These effects, however, are outweighed by the influence of an increase in θ^* and $(\sigma^*)^2$. The expected rate of depreciation seems to converge to a value which can be approximated by $e^{(\theta-\theta^*)\cdot 0.01}-1$. This can be easily interpreted if one considers that the expectation in the long-run can be approximated by $e^{\theta\cdot 0.01}-1$. This example, however, shows that the expected depreciation rates calculated using the results of Chapter 4 and Chapter 5 develop totally differently. Finally, the different development of the two expected depreciation rates is inconsistent with the expected equality stated by the expectations hypothesis.

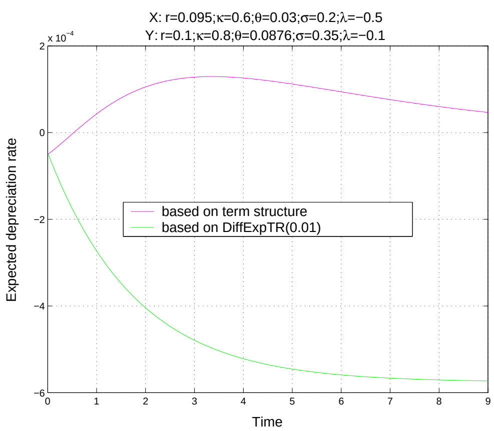
The phenomenon of an appreciation in the beginning, followed by a depreciation is not unknown at all. One can often observe the exchange rate to overshoot and undershoot its final level respectively. One explanation is that the equilibrium on the foreign exchange rate market is reached faster than one on the market for goods. For further information see Krugman and Obstfeld [11].



(a) Term Structure



(b) Expected Exchange Rate



(c) Expected Depreciation Rate

Figure 7.6: Example 6

7.2 New Expectations and the Adjustment of the Spot Exchange Rate

In this section, I present two examples of how observed values of the spot rate at time $s = 1$ influence the spot exchange rate and the expectations with regard to the results of Chapter 6. I chose the economies presented in Example 5. Initially, the spot rates at time $t = 0$ were 5.5% in both countries. Figure 7.7 displays the results when at time $s = 1$ one can observe the spot rate to be risen to 6% in country Y, while it has not changed in country X. As the other variables describing both economies have not changed, the long-term yields (3.52) of both economies have not changed either.

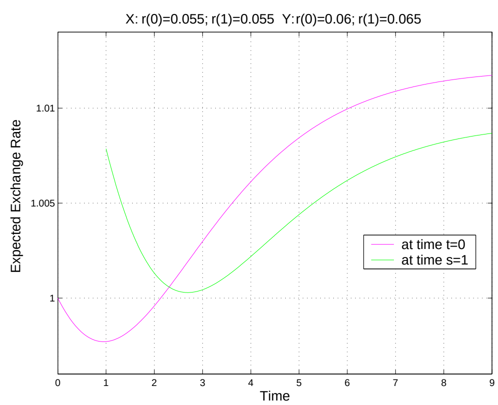


Figure 7.7: Example 7

The structure of the expectations, however, differs from the initial structure, because different values of the spot rate influence the prices of the bond. Equation (3.47) shows that changes of r result in changes of the price. Furthermore, the different values of the spot exchange rate at time $s = 1$ also influence the expectations as (4.5) shows. One would expect an exchange rate that is above its initial level, as the spot rate in country Y increased to a higher level. Consequently, one would expect a higher demand for the currency of country Y which results in a depreciation of the currency of country X and thus to an increase in the exchange rate. Obviously, the calculated spot exchange rate using (6.9) does meet this expectation. Additionally, it ensures that the squared weighted differences of the expectations formed at time $t = 0$ and those formed at time $s = 1$ are minimized. To ensure that the expectations almost stay the same, the spot exchange rate adjusts to $E_1 = 1.0014$.

On the contrary, an increase in the spot rate in country X leads to an appreciation of its currency. Figure 7.8 and the spot exchange rate of $E_1 = 0.9871$ confirms that.

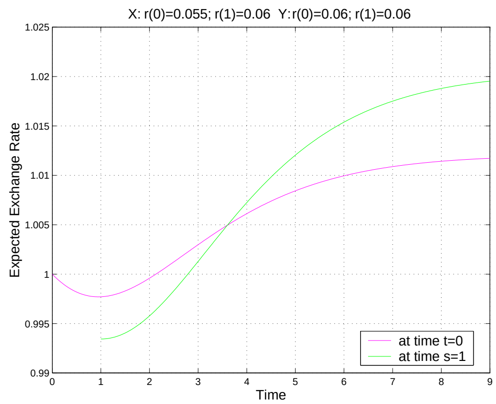


Figure 7.8: Example 8

Chapter 8

Summary

In this thesis, the structure of the expected exchange rates was investigated. The emphasis was put on the influence of the fundamental factors of the economies on the expectations.

Based on the term structures resulting from the model of Cox, Ingersoll, and Ross [6] I constructed a path of expected exchange rates under certain assumptions. Various investment opportunities of the agents based on the two different term structures led to an expectation of the future development of the spot exchange rate. Additionally, a further expectation was formed which was based on the assumptions that there are individuals who expect a certain exchange rate at a future time resulting from a possible repeated investment in short-term bonds. An explicit solution of that expectation dependent on the fundamental factors could be given in Chapter 5. The behavior of this expectation dependent on changes in the variables, except for the variable describing the volatility of the spot rate, seems to be consistent with the behavior of the term structure stated in the paper [6]. Consequently, it was easy to find economic reasonable interpretations for the observed behavior. The lack of a possible explanation for the influence of the variance of the current interest rate and for the significance of the length of the period on the expected future one period total return may indicate deeper and further connections. Further investigation, which would be beyond the scope of that thesis, could lead to further knowledge. On the contrary, the so far unexplained behavior could also speak for an unusable approach.

With the help of these expectations I calculated another exchange rate expectation with regard to future times. Both investigated expected depreciation rates partly show similar, almost identical behavior, however, partly absolutely different behavior. Nevertheless, all examples presented in Chapter 7 show reasonable structures of the expectations. This poses the question of which path of the expected exchange rate is more meaningful and reasonable respectively. I hold the view that the question cannot be answered easily as each argumentation underlying the particular expectation is reasonable. Consequently, it is hard to be in favor of one of the paths. It can rather be assumed that in the long-run both expectations are not meaningful at all as the influence of the different price levels, the inflation rate and the trading of goods are neglected. Within the framework of the purchasing power parity the influence of price levels gains more importance in the long-run. As a result, the presented approach may not provide reasonable statements with regard to the expected exchange rates in the long-run.

In addition to that, I presented a possibility of determining the spot exchange rate taking the whole path of expected exchange rates into consideration. This represents an extension to the more simple models explaining the influence of the expectation on the current value. Although an analytical solution could not be obtained, a way to calculate the spot exchange rate using numerical integration methods was shown.

Appendix A

Solving a Stochastic Differential Equation: an example

The following example will outline how the definition of the Itô Integral and of the Itô formula can be used to calculate solutions for stochastic differential equations.

We want to solve the following SDE:

$$dX_t = \frac{1}{2}X_t dt + X_t dB_t. \tag{A.1}$$

Choosing $g(t,x) = ln(x)$ and using Itô's formula leads to:

$$\begin{aligned} d(ln(X_t)) &= \frac{1}{X_t}dX_t + \frac{1}{2}\Big(\frac{-1}{X_t^2}\Big)(dX_t)^2 \\ &= \frac{1}{X_t}dX_t + \frac{1}{2}\Big(\frac{-1}{X_t^2}\Big)(\frac{1}{2}X_t dt + X_t dB_t)^2 \\ &= \frac{1}{X_t}dX_t - \frac{1}{2}\frac{1}{X_t^2}X_t^2 dt \\ &= \frac{1}{X_t}dX_t - \frac{1}{2}dt. \end{aligned} \tag{A.2}$$

Rewriting (A.1) as

$$\frac{dX_t}{X_t} = \frac{1}{2}dt + dB_t,$$

it follows together with (A.2) that:

$$\begin{aligned} d(ln(X_t)) + \frac{1}{2}dt &= \frac{1}{2}dt + dB_t \\ d(ln(X_t)) &= dB_t \\ \int_0^t d(ln(X_s)) &= B_t \end{aligned}$$

$$
\begin{aligned}
ln\left(\frac{X_t}{X_0}\right) &= B_t \\
\frac{X_t}{X_0} &= e^{B_t} \\
X_t &= X_0 e^{B_t}.
\end{aligned}
$$

Appendix B

Further Technical Notes

B.1 The paper CIRI

Solution to the stochastic control problem:

$$\begin{pmatrix} dW \\ dY \end{pmatrix} = \begin{pmatrix} W\mu(W) \\ \mu \end{pmatrix} dt + \begin{pmatrix} W\Sigma_{j=1}^{n+k} q_j \\ S \end{pmatrix} d\omega.$$

This equation can be interpreted as (2.7), where $b = \begin{pmatrix} W\mu(W) \\ \mu \end{pmatrix}$ and $\sigma = \begin{pmatrix} W\Sigma_{j=1}^{n+k} q_j \\ S \end{pmatrix}$.

Here $\nu(t, W, Y)$ is the parameter whose value we can choose to control the process. V is the Borel set U mentioned in Section 2.2.3. The function to be maximized can be expressed with certain values for f^ν and g as in (2.8). In this case, there is $f^\nu = U(\nu(s), Y(s), s)$ and $g = 0$.

The indirect utility function $J(W, Y, t)$ can be interpreted as the function Φ in (2.9).

If we further consider that
$W\mu(W)dt \equiv \Big[\sum_{i=1}^n a_i W(\alpha_i - r) + \sum_{i=1}^k b_i W(\beta_i - r) + rW - C\Big] dt$,
$W\sum_{j=1}^{n+k} q_j d\omega_j \equiv \sum_{i=1}^n a_i W\Big(\sum_{j=1}^{n+k} g_{ij} d\omega_j\Big) + \sum_{i=1}^k b_i W\Big(\sum_{j=1}^{n+k} h_{ij} d\omega_j\Big)$, and
$a_i \geq 0, C \geq 0$, the theorem of Hamilton-Jacobi-Bellman, that is calculating $\sup_{\nu \in V}\{L^\nu J + U\}$ (see (2.11)), leads to the equations (3.7)-(3.11).

B.2 The paper CIRII

to (3.26):

$$J(W,Y,t) = f(Y,t)U(W,t) + g(Y,t)$$

$$\begin{aligned}
J(W,Y,t)_W &= f(Y,t)U(W,t)_W \\
J(W,Y,t)_{WW} &= f(Y,t)U(W,t)_{WW} \\
U(W,t)_W &= \Big\{e^{-\rho s}\Big[\frac{W^\gamma - 1}{\gamma}\Big]\Big\}_W \\
&= e^{-\rho s}W^{\gamma-1} \\
U(W,t)_{WW} &= \Big\{e^{-\rho s}\Big[\frac{W^\gamma - 1}{\gamma}\Big]\Big\}_{WW} \\
&= e^{-\rho s}(\gamma-1)W^{\gamma-2}
\end{aligned}$$

$$\Rightarrow \quad \begin{aligned}
\frac{-WJ(W,Y,t)_{WW}}{J(W,Y,t)_W} &= \frac{-We^{-\rho s}(\gamma-1)W^{\gamma-2}}{e^{-\rho s}W^{\gamma-1}} \\
&= 1-\gamma
\end{aligned}$$

to (3.27):

$$J(W,Y,t) = f(Y,t)U(W,t) + g(Y,t)$$

$$J_W = f(Y,t)U(W,t)_W$$

$$\begin{aligned}
J_{WY} &= f(Y,t)_Y U(W,t)_W \\
J_{WY} &= f(Y,t)\frac{J_W}{f(Y,t)}
\end{aligned}$$

$$\Rightarrow \quad \frac{-J_{WY}}{J_W} = \frac{-f_Y}{f}$$

to (3.28):

Can be calculated easily if one consider that for $\gamma = 0$

$$\begin{aligned}
WJ_{WW} &= -J_W \\
f_Y &= 0
\end{aligned}$$

as $f(Y,t) = \frac{1-e^{-\rho(t^{'}-t)}}{\rho}$ for $\gamma = 0$.

to (3.32):

Note that $r = \frac{\lambda^*}{WJ_W}$ as in (3.12). With (see (3.6))

$$\Psi = \alpha W J_W + GG^T a^* W^2 J_{WW} + GS^T W J_{WY} - \lambda^* 1 = 0$$

there is

$$\begin{aligned}
\frac{a^{*T}\Psi}{WJ_W} &= a^{*T}\alpha + a^{*T}GG^T a^* \frac{WJ_{WW}}{J_W} + a^{*T}GS^T\frac{J_{WY}}{J_W} - \frac{a^{*T}\lambda^*}{WJ_W} \\
&= a^{*T}\Big(\alpha - \frac{\lambda^*}{WJ_W}\Big) - a^{*T}GG^T a^* \\
&= a^{*T}\Big(\alpha - \frac{\lambda^*}{WJ_W} - GG^T a^*\Big)
\end{aligned}$$

$$\frac{a^{*T}\Psi}{WJ_W} = 0 \Leftrightarrow \quad \Big(\alpha - \frac{\lambda^*}{WJ_W} - GG^T a^*\Big) = 0.$$

That is

$$a^* = (GG^T)^{-1}\Big(\alpha - \frac{\lambda^*}{WJ_W}\Big).$$

Comparing that equation with (3.29) leads to (3.32).

to (3.42):

Proof of the mean value using (2.5):

$$\begin{aligned}
dr &= \kappa(\theta - r)dt + \sigma\sqrt{r}dz_1 \\
r(s) &= r(t) + \int_t^s \kappa(\theta - r(i))di + \int_t^s \sigma\sqrt{r(i)}dz_1(i) \\
E(r(s)) &= E(r(t)) + \int_t^s E(\kappa(\theta - r(i)))di + E(\int_t^s \sigma\sqrt{r(i)}dz_1(i)) \\
&= E(r(t)) + \int_t^s \kappa\theta di - \int_t^s \kappa E(r(i))di + 0 \\
\dot{E}(r(s)) &= \kappa\theta - \kappa E(r(s)), \quad E(r(t)) = r(t).
\end{aligned}$$

A propagator[1] satisfies the following equation

$$P(t;\tau) := e^{\int_\tau^t A(s)ds} \tag{B.1}$$

[1]For further information refer to [16].

for linear differential equations of the form:

$$\begin{aligned} x' &= A(t)x + B(t) \\ x(0) &= x_0. \end{aligned}$$

Hence, the linear differential equation can be solved as follows:

$$\begin{aligned} E(r(s)) &= r(t)e^{\int_t^s -\kappa ds} + \int_t^s P(s,\tau)\kappa\theta d\tau \\ &= r(t)e^{-\kappa(s-t)} + \int_t^s e^{\int_\tau^s -\kappa ds}\kappa\theta d\tau \\ &= r(t)e^{-\kappa(s-t)} + \int_t^s e^{-\kappa(s-\tau)}\kappa\theta d\tau \\ &= r(t)e^{-\kappa(s-t)} + \kappa\theta e^{-\kappa s}\int_t^s e^{\kappa\tau} d\tau \\ &= r(t)e^{-\kappa(s-t)} + \kappa\theta e^{-\kappa s}[e^{\kappa\tau}\frac{1}{\kappa}]_t^s \\ &= r(t)e^{-\kappa(s-t)} + \kappa\theta e^{-\kappa s}(e^{\kappa s}\frac{1}{K} - e^{\kappa t}\frac{1}{\kappa}) \\ &= r(t)e^{-\kappa(s-t)} + \theta(1 - e^{-\kappa(s-t)}). \end{aligned}$$

to (3.45):

Using (3.29), there is

$$\begin{aligned} \phi_Y &= a^{*T}\Sigma Y \\ &= \Big[(GG^T)^{-1}\alpha + \Big(\frac{1 - 1^T(GG^T)^{-1}\alpha}{1^T(GG^T)^{-1}1}\Big)(GG^T)^{-1}1\Big]^T \Sigma Y \\ &= \Big[\Omega^{-1}\hat{\alpha} + \Big(\frac{1 - 1^T\Omega^{-1}\hat{\alpha}}{1^T\frac{\Omega^{-1}}{Y}1}1\Big)\frac{\Omega^{-1}}{Y}1\Big]^T \Sigma Y \\ &= \Big[\hat{\alpha}^T\Omega^{-1}\Sigma + \Big(\frac{1 - 1^T\Omega^{-1}\hat{\alpha}}{1^T\Omega^{-1}1}\Big)1^T\Omega^{-1}\Sigma\Big]Y. \end{aligned}$$

B.3 The Expected Exchange Rates

to (4.4):

With $R(r,t,T) = (rB(t,T) - ln(A(t,T)))/(T-t)$ and $P(r,t,T) = A(t,T)e^{-rB(t,T)}$ the proof is as follows:

$$R(r,t,T)^{TR} = e^{R(r,t,T)\cdot(T-t)} - 1$$

$$\begin{aligned} &= \frac{e^{rB(t,T)}}{A} - 1 \\ &= \frac{1}{A(t,T)e^{-rB(t,T)}} - 1 \\ &= \frac{1}{P(r,t,T)} - 1. \end{aligned}$$

B.4 The Liquidity Preference Hypothesis

to (5.3):

Using a sufficient small Δ such that

$$\begin{aligned} e^{\gamma\Delta} &\approx 1 + \gamma\Delta \\ (\gamma+\kappa+\lambda)\Delta + 2 &\approx 2 \\ ln\Big(\frac{c}{c-\Delta}\Big) &\approx \frac{c}{c-\Delta} - 1 \\ \frac{\Delta}{c-\Delta} &\approx \frac{\Delta}{c} \end{aligned}$$

Consequently,

$$\begin{aligned} B(s,s+\Delta) &= \frac{2(e^{\gamma\Delta}-1)}{(\gamma+\kappa+\lambda)(e^{\gamma\Delta}-1)+2\gamma} \\ &\approx \frac{2\Delta\gamma}{(\gamma+\kappa+\lambda)\Delta\gamma+2\gamma} \\ &= \frac{2\Delta}{(\gamma+\kappa+\lambda)\Delta+2} \\ &\approx \frac{2\Delta}{2} \\ &= \Delta \end{aligned}$$

and analogous to that

$$A(s,s+\Delta) \approx 1.$$

Hence,

$$E(R(r(s),s,s+\Delta)^{TR}) =$$

$$\begin{aligned} &= \frac{1}{A(s,s+\Delta)}\Big(\frac{c}{c-B(s,s+\Delta)}\Big)^{q+1} e^{\frac{uB(s,s+\Delta)}{c-B(s,s+\Delta)}} - 1 \\ &\approx \Big(\frac{c}{c-\Delta}\Big)^{q+1} e^{\frac{u\Delta}{c-\Delta}} - 1 \end{aligned}$$

$$
\begin{aligned}
&= e^{\frac{2\kappa\theta}{\sigma^2} ln(\frac{c}{c-\Delta})} e^{\frac{cr(t)e^{-\kappa(s-t)}\Delta}{c-\Delta}} - 1 \\
&\approx e^{\frac{2\kappa\theta}{\sigma^2}\left(\frac{c}{c-\Delta}-1\right)} e^{r(t)e^{-\kappa(s-t)}\Delta} - 1 \\
&= e^{\frac{2\kappa\theta}{\sigma^2}\frac{\Delta}{c-\Delta}} e^{r(t)e^{-\kappa(s-t)}\Delta} - 1 \\
&\approx e^{\frac{2\kappa\theta}{\sigma^2}\frac{\Delta\sigma^2(1-e^{-\kappa(s-t)})}{2\kappa}} e^{r(t)e^{-\kappa(s-t)}\Delta} - 1 \\
&= e^{r(t)e^{-\kappa(s-t)}+\theta(1-e^{-\kappa(s-t)})} - 1 \\
&= e^{E(r(s)|r(t))} - 1.
\end{aligned}
$$

B.5 Calculation of the ExpTR(Δ)

At this point I present a possible alternative of calculating the expectation.

The spot rate needs to satisfy the following stochastic differential equation:

$$dr = \kappa(\theta - r)dt + \sigma\sqrt{r}dz_1.$$

Calculating the expected interest rate differentials for one period bonds results in constant $A(s, s+\Delta)$ and $B(s, s+\Delta)$, denoted by $\bar{A}$ and $\bar{B}$.

Using Itô's formula and (2.5) we can calculate the stochastic differential equation for $Y_t = g(t, r(t)) = e^{r(t)\bar{B}-ln(\bar{A})}$:

$$
\begin{aligned}
dY(s) &= \frac{\bar{B}}{\bar{A}} e^{r(s)\bar{B}} dr_s + \frac{1}{2}\frac{\bar{B}^2}{\bar{A}} e^{r(s)\bar{B}} ds \\
Y(s) &= e^{r(t)\bar{B}-ln(\bar{A})} + \int_t^s \frac{1}{2}\frac{e^{r(i)\bar{B}}\bar{B}^2}{\bar{A}} di + \int_t^s \frac{e^{r(i)\bar{B}}}{\bar{A}} \bar{B} dr(i) \\
&= e^{r(t)\bar{B}-ln(\bar{A})} + \int_t^s \frac{1}{2}\frac{e^{r(i)\bar{B}}\bar{B}^2}{\bar{A}} di + \int_t^s \frac{e^{r(i)\bar{B}}\bar{B}}{\bar{A}} \kappa(\theta - r(i)) di + \int_t^s \frac{e^{r(i)\bar{B}}\bar{B}}{\bar{A}} \sigma\sqrt{r(i)} dz_1(i) \\
E(Y(s)) &= E(e^{r(t)\bar{B}-ln(\bar{A})}) + \frac{1}{2}\frac{\bar{B}^2}{\bar{A}} \int_t^s E(e^{r(i)\bar{B}}) di + \frac{\bar{B}}{\bar{A}} \int_t^s E(e^{r(i)\bar{B}} \kappa(\theta - r(i))) di \\
\dot{E}(Y(s)) &= \frac{1}{2}\frac{\bar{B}^2}{\bar{A}} E(e^{r(s)\bar{B}}) + \frac{\bar{B}}{\bar{A}} E(e^{r(s)\bar{B}}\kappa\theta) + \frac{\bar{B}}{\bar{A}} E(-\kappa r(s) e^{r(s)\bar{B}}) \\
&= E(Y(s))(\frac{1}{2}\bar{B}^2 + \bar{B}\kappa\theta) + E(Y(s)(-\kappa r(s)))\bar{B}.
\end{aligned}
$$

As one can see, this approach, however, does not lead to a linear differential equation. Hence, a solution may not be trivial.

$\bar{u} > 0$, because with $\kappa > 0$, $\sigma^2 > 0$, and $s > 0$ there is $e^{-\kappa(s-t)} < 1$. Hence, $c > 0$. Consequently, $u > 0$. Hence, $\bar{u} > 0$.

B.6 Characteristics of the ExpTR(Δ)

To show the continuity of the partial derivatives the following facts are helpful:

$$2\kappa^2 \quad > \quad |2\kappa\lambda|,$$

as $\kappa > |\lambda|$.

Furthermore,

$$2\kappa\gamma > 2\sigma^2.$$

With $\sigma^2 < -2\kappa\lambda$ the proof is as follows:

$$\begin{aligned}
2\kappa\sqrt{(\kappa+\lambda)^2+2\sigma^2} &> 2\sigma^2 \quad \Leftrightarrow \\
\kappa\sqrt{(\kappa+\lambda)^2+2\sigma^2} &> \sigma^2 \quad \Leftrightarrow \\
\kappa^2[(\kappa+\lambda)^2+2\sigma^2] &> \sigma^4 \quad \Leftrightarrow \\
\kappa^2(\kappa+\lambda)^2+\sigma^2(2\kappa^2-\sigma^2) &> 0 \quad \Leftrightarrow \\
\kappa^2(\kappa+\lambda)^2+\sigma^2(2\kappa^2+2\kappa\lambda) &> 0.
\end{aligned}$$

Additionally,

$$\gamma \quad > \quad \kappa+\lambda.$$

There is

$$(\gamma+\kappa+\lambda)(e^{\gamma(T-t)}-1)+2\gamma > 0,$$

as

$$\begin{aligned}
(\gamma+\kappa+\lambda)(e^{\gamma(T-t)}-1)+2\gamma &> 0 \quad \Leftrightarrow \\
(\gamma+\kappa+\lambda)e^{\gamma(T-t)}-(\gamma+\kappa+\lambda)+2\gamma &> 0 \quad \Leftrightarrow \\
(\gamma+\kappa+\lambda)e^{\gamma(T-t)}+\gamma-\kappa-\lambda &> 0
\end{aligned}$$

and

$$\begin{aligned}
\gamma-\kappa-\lambda &> 0 \quad \Leftrightarrow \\
\sqrt{(\kappa+\lambda)^2+2\sigma^2} &> \kappa+\lambda.
\end{aligned}$$

Appendix C

MATLAB Source Files

I implemented all algorithms using MATLAB. Moreover, all figures were plotted using MATLAB.

<u>Source file characterize.m</u>:

```
%This m-file will create the values of the two different countries
%and/or two different economic situations in one country at current
%time t respectively
clear all;

%COUNTRY X
r_X=0.055; kappa=0.6; theta=0.03; sigma=0.4; lambda=-0.5;
gamma=((kappa+lambda)^2+2*(sigma^2))^0.5;
ValX=[r_X,kappa,theta,sigma,lambda]; save ValX.mat ValX; clear
all;

%COUNTRY Y (or second situation)
r_Y=0.06; kappa=0.8; theta=0.0314; sigma=0.4; lambda=-0.5;
gamma=((kappa+lambda)^2+2*(sigma^2))^0.5;
ValY=[r_Y,kappa,theta,sigma,lambda]; save ValY.mat ValY; clear
all;

%Time and exchange rate
t=0; lasttime=9; ER_X_Y=1; ER_X_Y_at_t=[t,lasttime,ER_X_Y]; save
ER_X_Y_at_t.mat ER_X_Y_at_t; clear all;
```

Source file termstructure.m:

```
%This m-file is used for calculation and investigation
%of the term structure of yields
clear all;

load ER_X_Y_at_t.mat
%Current exchange rate
ER_X_Y=ER_X_Y_at_t(3);
%Current time
t=ER_X_Y_at_t(1); lasttime=ER_X_Y_at_t(2);

%COUNTRY X
%COUNTRY X with following values
load ValX.mat r=ValX(1); kappa=ValX(2); theta=ValX(3);
sigma=ValX(4); lambda=ValX(5);
gamma=((kappa+lambda)^2+2*(sigma^2))^0.5;
%Long term value of the yield
Rl=(2*kappa*theta)/(gamma+kappa+lambda)
%Critical value, which determines whether or not
%the term structure is humped
Rg=(kappa*theta)/(kappa+lambda) phi1=gamma;
phi2=(kappa+lambda+gamma)*0.5; phi3=(2*kappa*theta)/(sigma^2);

%This part calculates the term structure of default-free
%discount bonds according to the paper of CIRII
%Yield converges to the spot rate as maturity nears:
%Tau==Null
Rrt(1)=r; i=1; for Tau=0.01:0.01:lasttime-t
    i=i+1;
    N=phi2*(exp(phi1*(Tau))-1)+phi1;
    At=(phi1*exp(phi2*(Tau))/N)^phi3;
    Bt=(exp(phi1*(Tau))-1)/N;
    if At~=0
        Rrt(i)=(-log(At)+r*Bt)/(Tau);
    else
        Rrt(i)=NaN;
    end
end
```

```
Values=[t:0.01:lasttime;Rrt]; visu4=figure;
plot(Values(1,:),Values(2,:));hold on;

clear kappa theta sigma lambda gamma phi1 phi2
phi3 r Rrt Prt i;

%COUNTRY Y
%COUNTRY Y with following values
load ValY.mat r=ValY(1); kappa=ValY(2); theta=ValY(3);
sigma=ValY(4); lambda=ValY(5);
gamma=((kappa+lambda)^2+2*(sigma^2))^0.5;
%Long term value of the yield
Rl=(2*kappa*theta)/(gamma+kappa+lambda)
%Critical value, which determines whether or not
%the term structure is humped
Rg=(kappa*theta)/(kappa+lambda) phi1=gamma;
phi2=(kappa+lambda+gamma)*0.5; phi3=(2*kappa*theta)/(sigma^2);

%This part calculates the term structure of default-free discount
%bonds according to the paper of CIRII
%Yield converges to the spot rate as maturity nears: Tau==Null
Rrt(1)=r; i=1; for Tau=0.01:0.01:lasttime-t
    i=i+1;
    N=phi2*(exp(phi1*(Tau))-1)+phi1;
    At=(phi1*exp(phi2*(Tau))/N)^phi3;
    Bt=(exp(phi1*(Tau))-1)/N;
    if At~=0
        Rrt(i)=(-log(At)+r*Bt)/(Tau);
    else
        Rrt(i)=NaN;
    end
end

Values=[t:0.01:lasttime;Rrt];
plot(Values(1,:),Values(2,:),'r');hold off; title(['VAL ONE:
',texlabel('r='),mat2str(ValX(1)),';',texlabel('kappa='),
mat2str(ValX(2)),';',texlabel('theta='),mat2str(ValX(3)),';',
texlabel('sigma='),mat2str(ValX(4)),';',texlabel('lambda='),
mat2str(ValX(5)),' VAL TWO:
',texlabel('r='),mat2str(ValY(1)),';',texlabel('kappa='),
```

```
mat2str(ValY(2)),';',texlabel('theta='),mat2str(ValY(3)),';',
texlabel('sigma='),mat2str(ValY(4)),';',texlabel('lambda='),
mat2str(ValY(5))],'Fontsize',11);legend([texlabel('kappa='),
mat2str(ValX(2))],[texlabel('kappa='),mat2str(ValY(2))],0);
xlabel('Time-to-maturity');ylabel('Yield-to-maturity');grid
on;

%SAVING OF PLOT with counting number i
i=9; saveas(visu4,['termstructure',mat2str(i),'.eps']);
saveas(visu4,['termstructure',mat2str(i),'.fig']);

clear all;
```

Source file exptr.m:

```
%This m-file is used for calculation and investigation
%of the expected total return of zero bonds with
%maturity=lag at future time s
clear all;

load ER_X_Y_at_t.mat
%Current exchange rate
ER_X_Y=ER_X_Y_at_t(3);
%Current time
t=ER_X_Y_at_t(1); lasttime=ER_X_Y_at_t(2);

lag=0.01;

%COUNTRY X
%COUNTRY X with following values
load ValX.mat r=ValX(1); kappa=ValX(2); theta=ValX(3);
sigma=ValX(4); lambda=ValX(5);
gamma=((kappa+lambda)^2+2*(sigma^2))^0.5;
%Long term value of the yield
Rl=(2*kappa*theta)/(gamma+kappa+lambda);
%Critical value, which determines whether or not
%the term structure is humped
Rg=(kappa*theta)/(kappa+lambda); phi1=gamma;
phi2=(kappa+lambda+gamma)*0.5; phi3=(2*kappa*theta)/(sigma^2);

%These values can be chosen constant, cause we only
%investigate bonds with maturity=lag
N=phi2*(exp(phi1*(lag))-1)+phi1; At=(phi1*exp(phi2*(lag))/N)^phi3;
Bt=(exp(phi1*(lag))-1)/N;

%Value in the long-run
Yl=(1/At)*((2*kappa/sigma^2)/((2*kappa/sigma^2)-Bt))^
(2*kappa*theta/sigma^2)-1

%This part calculates the expectation of the TR
%when reinvesting every period
%The yield for a zero bond with time to maturity = lag
%at the current time t=0
```

```
if At~=0
    Rrt=(-log(At)+r*Bt)/(lag);
else
    Rrt=NaN;
end

%The expected total return for a zero bond with
%maturity=0.01 at current time t can be calculated
%as follows:
RTR(1)=exp(Rrt*lag)-1; i=1; for s=lag+t:lag:lasttime-0.01
    i=i+1;
    c=(2.*kappa)./(sigma.^2.*(1-exp(-kappa.*(s-t))));
    u=c.*r.*exp(-kappa.*(s-t));
    q=(2.*kappa.*theta)./(sigma.^2)-1;
    RTR(i)=(1/At)*(c/(c-Bt))^(q+1)*exp((u*Bt)/(c-Bt))-1;
end

visu2=figure; plot(t:lag:lasttime-0.01,RTR,'b');hold on;

clear kappa theta sigma lambda gamma phi1 phi2 phi3
r Rrt i RTR s
c u q Yl At Bt;

%COUNTRY Y
%COUNTRY Y with following values
load ValY.mat r=ValY(1); kappa=ValY(2); theta=ValY(3);
sigma=ValY(4); lambda=ValY(5);
gamma=((kappa+lambda)^2+2*(sigma^2))^0.5;
%Long term value of the yield
Rl=(2*kappa*theta)/(gamma+kappa+lambda);
%Critical value, which determines whether or not
%the term structure is humped
Rg=(kappa*theta)/(kappa+lambda); phi1=gamma;
phi2=(kappa+lambda+gamma)*0.5; phi3=(2*kappa*theta)/(sigma^2);

%These values can be chosen constant, cause we only
%investigate bonds with maturity=lag
N=phi2*(exp(phi1*(lag))-1)+phi1; At=(phi1*exp(phi2*(lag))/N)^phi3;
Bt=(exp(phi1*(lag))-1)/N;
```

```
%Value in the long-run%
Yl=(1/At)*((2*kappa/sigma^2)/((2*kappa/sigma^2)-Bt))^
(2*kappa*theta/sigma^2)-1

%This part calculates the expectation of the TR when
%reinvesting every period
%The yield for a zero bond with time to maturity = lag
% at the current time t=0
if At~=0
    Rrt=(-log(At)+r*Bt)/(lag);
else
    Rrt=NaN;
end

%The expected total return for a zero bond with
%maturity=0.01 at current time t can be calculated
% as follows:
RTR(1)=exp(Rrt*lag)-1; i=1; for s=lag+t:lag:lasttime-0.01
    i=i+1;
    c=(2*kappa)/(sigma^2*(1-exp(-kappa*(s-t))));
    u=c*r*exp(-kappa*(s-t));
    q=(2*kappa*theta)/(sigma^2)-1;
    RTR(i)=(1/At)*(c/(c-Bt))^(q+1)*exp((u*Bt)/(c-Bt))-1;
end

plot(t:lag:lasttime-0.01,RTR,'r'); title(['VAL ONE:
',texlabel('r='),mat2str(ValX(1)),';',texlabel('kappa='),
mat2str(ValX(2)),';',texlabel('theta='),mat2str(ValX(3)),';',
texlabel('sigma='),mat2str(ValX(4)),';',texlabel('lambda='),
mat2str(ValX(5)),' VAL TWO:
',texlabel('r='),mat2str(ValY(1)),';',texlabel('kappa='),
mat2str(ValY(2)),';',texlabel('theta='),mat2str(ValY(3)),';',
texlabel('sigma='),mat2str(ValY(4)),';',texlabel('lambda='),
mat2str(ValY(5))],'Fontsize',11);legend([texlabel('sigma^2='),
mat2str(ValX(4)^2)],[texlabel('sigma^2='),mat2str(ValY(4)^2)],0);
xlabel('Time
s');ylabel('Expected future one period total return');grid on;


%SAVING OF PLOT with counting number i
```

```
i=12; saveas(visu2,['Exp1TR',mat2str(i),'.eps']);
saveas(visu2,['Exp1TR',mat2str(i),'.fig']);

clear all;
```

Source file testliqpref.m:

```
%This m-file is used to test whether the investment in bonds with
%longer maturity pays off more than the repeated investment in bonds
%with maturity=0.01
clear all;

load ER_X_Y_at_t.mat
%Current exchange rate
ER_X_Y=ER_X_Y_at_t(3);
%Current time
t=ER_X_Y_at_t(1); lasttime=ER_X_Y_at_t(2);

%COUNTRY X
%COUNTRY X with following values
load ValX.mat r=ValX(1); kappa=ValX(2); theta=ValX(3);
sigma=ValX(4); lambda=ValX(5);
gamma=((kappa+lambda)^2+2*(sigma^2))^0.5;
%Long term value of the yield%
Rl=(2*kappa*theta)/(gamma+kappa+lambda);
%Critical value, which determines whether or not the
%term structure is humped
Rg=(kappa*theta)/(kappa+lambda); phi1=gamma;
phi2=(kappa+lambda+gamma)*0.5; phi3=(2*kappa*theta)/(sigma^2);

%This part calculates the term structure of default-free discount
%bonds according to the paper of CIRII
%Yield converges to the spot rate as maturity nears: Tau==Null
Rrt(1)=r; i=1; for Tau=0.01:0.01:lasttime-t
    i=i+1;
    N=phi2*(exp(phi1*(Tau))-1)+phi1;
    At=(phi1*exp(phi2*(Tau))/N)^phi3;
    Bt=(exp(phi1*(Tau))-1)/N;
    if At~=0
        Rrt(i)=(-log(At)+r*Bt)/(Tau);
        RTRl(i-1)=exp(Rrt(i)*Tau)-1;
    else
        Rrt(i)=NaN;
        RTRl(i-1)=NaN;
    end
```

```
end

%Plotting the effective yield when at time t=0 a zero bond
%with maturity Tau=[0.01:0.01:lasttime-t] is chosen
visu5=figure; Values=[0.01+t:0.01:lasttime;RTRl];
plot(Values(1,:),Values(2,:));hold on;axis tight

%These values can be chosen constant, cause we only investigate
%bonds with maturity=0.01
N=phi2*(exp(phi1*(0.01))-1)+phi1;
At=(phi1*exp(phi2*(0.01))/N)^phi3; Bt=(exp(phi1*(0.01))-1)/N;

%This part calculates the expectation of the total return
%when reinvesting every period
%The expected total return for a zero bond with maturity=0.01
%at current time t can be calculated as follows:
RTR(1)=exp(Rrt(2)*0.01)-1; i=1; for s=0.01+t:0.01:lasttime-0.01
    i=i+1;
    c=(2.*kappa)./(sigma.^2.*(1-exp(-kappa.*(s-t))));
    u=c.*r.*exp(-kappa.*(s-t));
    q=(2.*kappa.*theta)./(sigma.^2)-1;
    RTR(i)=(1/At)*(c/(c-Bt))^(q+1)*exp((u*Bt)/(c-Bt))-1;
end

%Calculation of the total return when reinvesting every period
i=1; for s=0.01+t:0.01:lasttime
    QMRTR(i)=1;
    for j=1:i
        QMRTR(i)=QMRTR(i)*(RTR(j)+1);
    end
    QMRTR(i)=QMRTR(i)-1;
    i=i+1;
end

%Plotting of the total return when reinvesting every period
plot(Values(1,:),QMRTR,'r');hold off;axis tight title('Country X:
Long-term bonds vs. repeated reinvestment'); legend('Total return
of long-term bond','Expected total return of repeated investment
in short-term bonds',0) xlabel('Date of
Maturity');ylabel('(Expected) Total return');grid on;
```

```
%COUNTRY Y
%COUNTRY Y with following values
load ValY.mat r=ValY(1); kappa=ValY(2); theta=ValY(3);
sigma=ValY(4); lambda=ValY(5);
gamma=((kappa+lambda)^2+2*(sigma^2))^0.5;
%Long term value of the yield%
Rl=(2*kappa*theta)/(gamma+kappa+lambda);
%Critical value, which determines whether or not the
%term structure is humped
Rg=(kappa*theta)/(kappa+lambda); phi1=gamma;
phi2=(kappa+lambda+gamma)*0.5; phi3=(2*kappa*theta)/(sigma^2);

%This part calculates the term structure of default-free discount
%bonds according to the paper of CIRII
%Yield converges to the spot rate as maturity nears: Tau==Null
Rrt(1)=r; i=1; for Tau=0.01:0.01:lasttime-t
    i=i+1;
    N=phi2*(exp(phi1*(Tau))-1)+phi1;
    At=(phi1*exp(phi2*(Tau))/N)^phi3;
    Bt=(exp(phi1*(Tau))-1)/N;
    if At~=0
        Rrt(i)=(-log(At)+r*Bt)/(Tau);
        RTRl(i-1)=exp(Rrt(i)*Tau)-1;
    else
        Rrt(i)=NaN;
        RTRl(i-1)=NaN;
    end
end

%Plotting the total return when at time t=0 a zero
%bond with maturity Tau=[0.01:0.01:lasttime-t] is chosen
visu6=figure; Values=[0.01+t:0.01:lasttime;RTRl];
plot(Values(1,:),Values(2,:));hold on;axis tight

%These values can be chosen constant, cause we only
%investigate bonds with maturity=0.01
N=phi2*(exp(phi1*(0.01))-1)+phi1;
At=(phi1*exp(phi2*(0.01))/N)^phi3; Bt=(exp(phi1*(0.01))-1)/N;
```

```
%This part calculates the expectation of the total return
%when reinvesting every period
%The expected total return for a zero bond with maturity=0.01
%at current time t can be calculated as follows:
RTR(1)=exp(Rrt(2)*0.01)-1; i=1; for s=0.01+t:0.01:lasttime-0.01
    i=i+1;
    c=(2.*kappa)./(sigma.^2.*(1-exp(-kappa.*(s-t))));
    u=c.*r.*exp(-kappa.*(s-t));
    q=(2.*kappa.*theta)./(sigma.^2)-1;
    RTR(i)=(1/At)*(c/(c-Bt))^(q+1)*exp((u*Bt)/(c-Bt))-1;
end

%Calculation of the effective yield when reinvesting every period
i=1; for s=0.01+t:0.01:lasttime
    QMRTR(i)=1;
    for j=1:i
        QMRTR(i)=QMRTR(i)*(RTR(j)+1);
    end
    QMRTR(i)=QMRTR(i)-1;
    i=i+1;
end

%Plotting of the total return when reinvesting every period
plot(Values(1,:),QMRTR,'r');hold off;axis tight title('Country Y:
Long-term bonds vs. repeated reinvestment'); legend('Total return
of long-term bond','Expected total return of repeated investment
in short-term bonds',0) xlabel('Date of
Maturity');ylabel('(Expected) Total return');grid on;

%SAVING OF PLOT with counting number i
i=1; saveas(visu5,['liqpref',mat2str(i),'-1.eps']);
saveas(visu6,['liqpref',mat2str(i),'-2.eps']);
saveas(visu5,['liqpref',mat2str(i),'-1.fig']);
saveas(visu6,['liqpref',mat2str(i),'-2.fig']);

clear all;
```

Source file exchangerates.m:

```
%This m-file is used to calculate the term structure and the
%resulting expectations of the development of the exchange rate
%These results are used to calculate the one period depreciation
%rates. These are compared to the one period depreciation
%rates following from the expected future one period total
%return differences
%The values of the variables need to be set in advance, e.g.
%using characterize.m
clear all;

load ER_X_Y_at_t.mat
%Current exchange rate
ER_X_Y=ER_X_Y_at_t(3);
%Current time
t=ER_X_Y_at_t(1); lasttime=ER_X_Y_at_t(2);

%COUNTRY X
%COUNTRY X with following values
load ValX.mat r_X=ValX(1); kappa=ValX(2); theta=ValX(3);
sigma=ValX(4); lambda=ValX(5);
gamma=((kappa+lambda)^2+2*(sigma^2))^0.5;
%Long term value of the yield
Rl_X=(2*kappa*theta)/(gamma+kappa+lambda)
%Critical value, which determines whether or not the
%term structure is humped
Rg_X=(kappa*theta)/(kappa+lambda) phi1=gamma;
phi2=(kappa+lambda+gamma)*0.5; phi3=(2*kappa*theta)/(sigma^2);

%Calculation of the term structure of COUNTRY X
%Yield converges to the spot rate as maturity nears: Tau==Null
Rrt(1)=r_X;
%Value of zero bond at maturity and effective interest rate
Prt(1)=1; i=1; for Tau=0.01:0.01:lasttime-t
    i=i+1;
    N=phi2*(exp(phi1*(Tau))-1)+phi1;
    At=(phi1*exp(phi2*(Tau))/N)^phi3;
    Bt=(exp(phi1*(Tau))-1)/N;
    Prt(i)=At*exp(-Bt*r_X);
```

```
    if At~=0
        Rrt(i)=(-log(At)+r_X*Bt)/(Tau);
    else
        Rrt(i)=NaN;
    end
end

N=phi2*(exp(phi1*(0.01))-1)+phi1;
At=(phi1*exp(phi2*(0.01))/N)^phi3; Bt=(exp(phi1*(0.01))-1)/N;
%Value in the long-run
Yl=(1/At)*((2*kappa/sigma^2)/((2*kappa/sigma^2)-Bt))
^(2*kappa*theta/sigma^2)-1
%The expected total return for a zero bond with maturity=0.01
%at current time t can be calculated as follows:
RTRX(1)=exp(Rrt(2)*0.01)-1; i=1; for s=0.01+t:0.01:lasttime-0.01
    i=i+1;
    c=(2.*kappa)./(sigma.^2.*(1-exp(-kappa.*(s-t))));
    u=c.*r_X.*exp(-kappa.*(s-t));
    q=(2.*kappa.*theta)./(sigma.^2)-1;
    RTRX(i)=(1/At)*(c/(c-Bt))^(q+1)*exp((u*Bt)/(c-Bt))-1;
end

%Saving
ValuesX=[t:0.01:lasttime;Rrt;Prt];
datafile=['X',mat2str(r_X),'_',mat2str(kappa),'_',mat2str(theta),
'_',mat2str(sigma),'_',mat2str(lambda),'.mat'];
save(datafile,'ValuesX');
phi_rX=[phi1,phi2,phi3,r_X,kappa,theta,sigma,lambda,gamma]; save
X.mat phi_rX;

clear kappa theta sigma lambda gamma phi1 phi2 phi3 i Tau Rrt Prt
r_X s c u q;

%COUNTRY Y
%COUNTRY Y with following values
load ValY.mat r_Y=ValY(1); kappa=ValY(2); theta=ValY(3);
sigma=ValY(4); lambda=ValY(5);
gamma=((kappa+lambda)^2+2*(sigma^2))^0.5;
%Long term value of the yield
Rl_Y=(2*kappa*theta)/(gamma+kappa+lambda)
```

```
%Critical value, which determines whether or not the
%term structure is humped
Rg_Y=(kappa*theta)/(kappa+lambda) phi1=gamma;
phi2=(kappa+lambda+gamma)*0.5; phi3=(2*kappa*theta)/(sigma^2);

%Calculation of the term structure of COUNTRY Y
%Yield converges to the spot rate as maturity nears: Tau==Null
Rrt(1)=r_Y;
%Value of zero bond at maturity and effective yield
Prt(1)=1; i=1; for Tau=0.01:0.01:lasttime-t
    i=i+1;
    N=phi2*(exp(phi1*(Tau))-1)+phi1;
    At=(phi1*exp(phi2*(Tau))/N)^phi3;
    Bt=(exp(phi1*(Tau))-1)/N;
    Prt(i)=At*exp(-Bt*r_Y);
    if At~=0
        Rrt(i)=(-log(At)+r_Y*Bt)/(Tau);
    else
        Rrt(i)=NaN;
    end
end

N=phi2*(exp(phi1*(0.01))-1)+phi1;
At=(phi1*exp(phi2*(0.01))/N)^phi3; Bt=(exp(phi1*(0.01))-1)/N;
%Value in the long-run
Yl=(1/At)*((2*kappa/sigma^2)/((2*kappa/sigma^2)-Bt))
^(2*kappa*theta/sigma^2)-1
%The expected total return for a zero bond with maturity=0.01
%at time current time t can be calculated as follows:
RTRY(1)=exp(Rrt(2)*0.01)-1; i=1; for s=0.01+t:0.01:lasttime-0.01
    i=i+1;
    c=(2.*kappa)./(sigma.^2.*(1-exp(-kappa.*(s-t))));
    u=c.*r_Y.*exp(-kappa.*(s-t));
    q=(2.*kappa.*theta)./(sigma.^2)-1;
    RTRY(i)=(1/At)*(c/(c-Bt))^(q+1)*exp((u*Bt)/(c-Bt))-1;
end

%Saving%
ValuesY=[t:0.01:lasttime;Rrt;Prt];
datafile=['Y',mat2str(r_Y),'_',mat2str(kappa),'_',mat2str(theta),
```

```
'_',mat2str(sigma),'_',mat2str(lambda),'.mat'];
save(datafile,'ValuesY');
phi_rY=[phi1,phi2,phi3,r_Y,kappa,theta,sigma,lambda,gamma]; save
Y.mat phi_rY;

clear kappa theta sigma lambda gamma phi1 phi2 phi3 i Tau Rrt Prt
r_Y s c u q;

%The expected exchange rates
%At current time the expectation equals the spot exchange rate
ExER(1)=ER_X_Y; i=1; for j=0.01+t:0.01:lasttime
    i=i+1;
    ExER(i)=((1/ValuesX(3,i))/(1/ValuesY(3,i)))*ER_X_Y;
end save ExER.mat ExER;

%Here the percentage change of the exchange rate from one
%period to another is calculated
%The values of the expected exchange rate are used
%f(i+1) defines the forward rate: i->i+1
i=0; for j=0.01+t:0.01:lasttime
    i=i+1;
    f(i+1)=(ExER(i+1)-ExER(i))/ExER(i);
end

%Here the percentage change of the exchange rate from one
%period to another is calculated
%The values of the differences of expected future one
%period total returns are used
l=length(RTRY); TRdiff=[RTRX(1:l)-RTRY(1:l)]./(1+RTRY(1:l));
l=length(f);

%Plot
visu1=figure;
plot(ValuesX(1,:),ValuesX(2,:),'r',ValuesY(1,:),ValuesY(2,:));axis
tight title(['X:
',texlabel('r='),mat2str(phi_rX(4)),';',texlabel('kappa='),
mat2str(phi_rX(5)),';',texlabel('theta='),mat2str(phi_rX(6)),';',
texlabel('sigma='),mat2str(phi_rX(7)),';',texlabel('lambda='),
mat2str(phi_rX(8)),' Y:
',texlabel('r='),mat2str(phi_rY(4)),';',texlabel('kappa='),
```

```
mat2str(phi_rY(5)),';',texlabel('theta='),mat2str(phi_rY(6)),';',
texlabel('sigma='),mat2str(phi_rY(7)),';',texlabel('lambda='),
mat2str(phi_rY(8))],'Fontsize',
11);legend('X','Y',0);xlabel('Time-to-maturity');
ylabel('Yield-to-maturity');grid on; visu2=figure;
plot(0+t:0.01:lasttime,ExER,'m');xlabel('Time');ylabel('Expected
Exchange Rate');axis tight title(['X:
',texlabel('r='),mat2str(phi_rX(4)),';',texlabel('kappa='),
mat2str(phi_rX(5)),';',texlabel('theta='),mat2str(phi_rX(6)),';',
texlabel('sigma='),mat2str(phi_rX(7)),';',texlabel('lambda='),
mat2str(phi_rX(8)),' Y:
',texlabel('r='),mat2str(phi_rY(4)),';',texlabel('kappa='),
mat2str(phi_rY(5)),';',texlabel('theta='),mat2str(phi_rY(6)),';',
texlabel('sigma='),mat2str(phi_rY(7)),';',texlabel('lambda='),
mat2str(phi_rY(8))],'Fontsize',11);grid on;
visu3=figure;plot(0.01+t:0.01:lasttime,f(2:l),'m');hold
on;plot(0.01+t:0.01:lasttime,TRdiff,'g');grid on; title(['X:
',texlabel('r='),mat2str(phi_rX(4)),';',texlabel('kappa='),
mat2str(phi_rX(5)),';',texlabel('theta='),mat2str(phi_rX(6)),';',
texlabel('sigma='),mat2str(phi_rX(7)),';',texlabel('lambda='),
mat2str(phi_rX(8)),' Y:
',texlabel('r='),mat2str(phi_rY(4)),';',texlabel('kappa='),
mat2str(phi_rY(5)),';',texlabel('theta='),mat2str(phi_rY(6)),';',
texlabel('sigma='),mat2str(phi_rY(7)),';',texlabel('lambda='),
mat2str(phi_rY(8))],'Fontsize',11);grid on; legend('based on term
structure','based on
DiffExpTR(0.01)',0);xlabel('Time');ylabel('Expected depreciation
rate');

%SAVING OF PLOT with counting number i
i=9; saveas(visu1,['Exrates',mat2str(i),'-1.fig']);
saveas(visu2,['Exrates',mat2str(i),'-2.fig']);
saveas(visu3,['Exrates',mat2str(i),'-3.fig']);
saveas(visu1,['Exrates',mat2str(i),'-1.eps']);
saveas(visu2,['Exrates',mat2str(i),'-2.eps']);
saveas(visu3,['Exrates',mat2str(i),'-3.eps']);

clear all;
```

Source file nextperiod.m:

```
%This m-file calculates the spot exchange rate at
%the next investigated time point
%It uses the m-files integral1.m and integral2.m
%If one wants to investigate another time point,
%integral1.m and integral2.m need to be adjusted
%The definite interval needs to be adjusted as well

%FIRST: Expectations at current time
clear all;

load ER_X_Y_at_t.mat
%Current exchange rate
ER_X_Y=ER_X_Y_at_t(3);
%Current time
t=ER_X_Y_at_t(1); lasttime=ER_X_Y_at_t(2);


%COUNTRY X
%COUNTRY X with following values
load ValX.mat r_X=ValX(1); kappa=ValX(2); theta=ValX(3);
sigma=ValX(4); lambda=ValX(5);
gamma=((kappa+lambda)^2+2*(sigma^2))^0.5; phi1=gamma;
phi2=(kappa+lambda+gamma)*0.5; phi3=(2*kappa*theta)/(sigma^2);

%Calculation of the term structure of COUNTRY X
%Yield converges to the spot rate as maturity nears: Tau==Null
Rrt(1)=r_X;
%Value of Zero Bond at maturity and effective interest rate
Prt(1)=1; i=1; for Tau=0.01:0.01:lasttime-t
    i=i+1;
    N=phi2*(exp(phi1*(Tau))-1)+phi1;
    At=(phi1*exp(phi2*(Tau))/N)^phi3;
    Bt=(exp(phi1*(Tau))-1)/N;
    Prt(i)=At*exp(-Bt*r_X);
    if At~=0
        Rrt(i)=(-log(At)+r_X*Bt)/(Tau);
    else
        Rrt(i)=NaN;
```

```
    end
end

%Saving
ValuesX=[t:0.01:lasttime;Rrt;Prt];
datafile=['X',mat2str(r_X),'_',mat2str(kappa),'_',mat2str(theta),
'_',mat2str(sigma),'_',mat2str(lambda),'.mat'];
save(datafile,'ValuesX');
phi_rX=[phi1,phi2,phi3,r_X,kappa,theta,sigma,lambda,gamma]; save
X.mat phi_rX;

ff=t; gg=r_X;

%Needed for SECOND part
phi1_1=gamma; phi2_1=(kappa+lambda+gamma)*0.5;
phi3_1=(2*kappa*theta)/(sigma^2); r1_1=r_X; r2_1=input('Please
enter the value (in \%) of the spot rate in Country X at time
t=0.01: '); r2_1=r2_1/100; r_X=r2_1; ValX(1)=r_X; save ValX.mat
ValX

%COUNTRY Y
%COUNTRY Y with following values
load ValY.mat r_Y=ValY(1); kappa=ValY(2); theta=ValY(3);
sigma=ValY(4); lambda=ValY(5);
gamma=((kappa+lambda)^2+2*(sigma^2))^0.5; phi1=gamma;
phi2=(kappa+lambda+gamma)*0.5; phi3=(2*kappa*theta)/(sigma^2);

%Calculation of the term structure of COUNTRY Y
%Yield converges to the spot rate as maturity nears: Tau==Null
Rrt(1)=r_Y;
%Value of Zero Bond at maturity and effective yield
Prt(1)=1;
%i is used for counting within the array
i=1; for Tau=0.01:0.01:lasttime-t
    i=i+1;
    N=phi2*(exp(phi1*(Tau))-1)+phi1;
    At=(phi1*exp(phi2*(Tau))/N)^phi3;
    Bt=(exp(phi1*(Tau))-1)/N;
    Prt(i)=At*exp(-Bt*r_Y);
    if At~=0
```

```
        Rrt(i)=(-log(At)+r_Y*Bt)/(Tau);
    else
        Rrt(i)=NaN;
    end
end

%Saving
ValuesY=[t:0.01:lasttime;Rrt;Prt];
datafile=['Y',mat2str(r_Y),'_',mat2str(kappa),'_',mat2str(theta),
'_',mat2str(sigma),'_',mat2str(lambda),'.mat'];
save(datafile,'ValuesY');
phi_rY=[phi1,phi2,phi3,r_Y,kappa,theta,sigma,lambda,gamma]; save
Y.mat phi_rY;

hh=r_Y;

%Needed for SECOND part
phi1_2=gamma; phi2_2=(kappa+lambda+gamma)*0.5;
phi3_2=(2*kappa*theta)/(sigma^2); r1_2=r_Y; r2_2=input('Please
enter the value (in \%) of the spot rate in Country Y at time
t=0.01: '); r2_2=r2_2/100; r_Y=r2_2; ValY(1)=r_Y; save ValY.mat
ValY;

%Calculation of the Expected Exchange Rate and plot%
%Equals the expected exchange rate at evaluation point 1
ExER(1)=ER_X_Y; i=1; for j=0.01+t:0.01:lasttime
    i=i+1;
    ExER(i)=((1/ValuesX(3,i))/(1/ValuesY(3,i)))*ER_X_Y;
end save ExER.mat ExER; visu1=figure;
plot(t:0.01:lasttime,ExER,'m');hold on;

%SECOND: Calculation of the spot exchange rate at next time
%x represents time T%
integrand1=['integral1(x,',mat2str(phi1_1),',',mat2str(phi2_1),
',',mat2str(phi3_1),',',mat2str(phi1_2),',',mat2str(phi2_2),',',
mat2str(phi3_2),',',mat2str(r1_1),',',mat2str(r2_1),',',
mat2str(r1_2),',',mat2str(r2_2),',',mat2str(ER_X_Y),')'];
integrand2=['integral2(x,',mat2str(phi1_1),',',mat2str(phi2_1),
',',mat2str(phi3_1),',',mat2str(phi1_2),',',mat2str(phi2_2),',',
mat2str(phi3_2),',',mat2str(r1_1),',',mat2str(r2_1),',',mat2str(r1_2),
```

```
',',mat2str(r2_2),')'];
E1=quad(integrand1,1,lasttime)/quad(integrand2,1,lasttime)
ER_X_Y_at_t=[1,lasttime,E1]; save ER_X_Y_at_t.mat ER_X_Y_at_t;

clear Prt Rrt ExER
%THIRD: Expectations at next time%
load ER_X_Y_at_t.mat
%Current exchange rate
ER_X_Y=ER_X_Y_at_t(3);
%Current time%
t=ER_X_Y_at_t(1); lasttime=ER_X_Y_at_t(2);

%COUNTRY X%
%COUNTRY X with following values
load ValX.mat r_X=ValX(1); kappa=ValX(2); theta=ValX(3);
sigma=ValX(4); lambda=ValX(5);
gamma=((kappa+lambda)^2+2*(sigma^2))^0.5; phi1=gamma;
phi2=(kappa+lambda+gamma)*0.5; phi3=(2*kappa*theta)/(sigma^2);

%Calculation of the term structure of COUNTRY X
%Yield converges to the spot rate as maturity nears: Tau==Null
Rrt(1)=r_X;
%Value of Zero Bond at maturity and effective interest rate
Prt(1)=1;
%i is used for counting within the array
i=1; for Tau=0.01:0.01:lasttime-t
    i=i+1;
    N=phi2*(exp(phi1*(Tau))-1)+phi1;
    At=(phi1*exp(phi2*(Tau))/N)^phi3;
    Bt=(exp(phi1*(Tau))-1)/N;
    Prt(i)=At*exp(-Bt*r_X);
    if At~=0
        Rrt(i)=(-log(At)+r_X*Bt)/(Tau);
    else
        Rrt(i)=NaN;
    end
end

%Saving%
ValuesX=[t:0.01:lasttime;Rrt;Prt];
```

```
datafile=['X',mat2str(r_X),'_',mat2str(kappa),'_',mat2str(theta),
'_',mat2str(sigma),'_',mat2str(lambda),'.mat'];
save(datafile,'ValuesX');
phi_rX=[phi1,phi2,phi3,r_X,kappa,theta,sigma,lambda,gamma]; save
X.mat phi_rX;

%COUNTRY Y
%COUNTRY Y with following values
load ValY.mat r_Y=ValY(1); kappa=ValY(2); theta=ValY(3);
sigma=ValY(4); lambda=ValY(5);
gamma=((kappa+lambda)^2+2*(sigma^2))^0.5; phi1=gamma;
phi2=(kappa+lambda+gamma)*0.5; phi3=(2*kappa*theta)/(sigma^2);

%Calculation of the term structure of COUNTRY Y
%Yield converges to the spot rate as maturity nears: Tau==Null
Rrt(1)=r_Y;
%Value of Zero Bond at maturity and effective yield
Prt(1)=1;
%i is used for counting within the array
i=1; for Tau=0.01:0.01:lasttime-t
    i=i+1;
    N=phi2*(exp(phi1*(Tau))-1)+phi1;
    At=(phi1*exp(phi2*(Tau))/N)^phi3;
    Bt=(exp(phi1*(Tau))-1)/N;
    Prt(i)=At*exp(-Bt*r_Y);
    if At~=0
        Rrt(i)=(-log(At)+r_Y*Bt)/(Tau);
    else
        Rrt(i)=NaN;
    end
end

%Saving%
ValuesY=[t:0.01:lasttime;Rrt;Prt];
datafile=['Y',mat2str(r_Y),'_',mat2str(kappa),'_',mat2str(theta),
'_',mat2str(sigma),'_',mat2str(lambda),'.mat'];
save(datafile,'ValuesY');
phi_rY=[phi1,phi2,phi3,r_Y,kappa,theta,sigma,lambda,gamma]; save
Y.mat phi_rY;
```

```
%Calculation of the Expected Exchange Rate and plot
%Equals the expected exchange rate at evaluation point 1
ExER(1)=ER_X_Y; i=1; for j=0.01+t:0.01:lasttime
    i=i+1;
    ExER(i)=((1/ValuesX(3,i))/(1/ValuesY(3,i)))*ER_X_Y;
end save ExER.mat ExER;

plot(t:0.01:lasttime,ExER,'g');xlabel('Time');ylabel('Expected
Exchange Rate'); title(['X:
',texlabel('r('),mat2str(ff),texlabel(')='),mat2str(gg),';
',texlabel('r('),mat2str(t),texlabel(')='),mat2str(r2_1),'  Y:
',texlabel('r('),mat2str(ff),texlabel(')='),mat2str(hh),';
',texlabel('r('),mat2str(t),texlabel(')='),mat2str(r2_2)],'Fontsize',
11);grid on; legend(['at time t=',mat2str(ff)],['at time
s=',mat2str(t)],0); clear all;
```

Source file integral1.m:

```
function
y=integral_1(x,phi1_1,phi2_1,phi3_1,phi1_2,phi2_2,phi3_2,
r1_1,r2_1,r1_2,r2_2,ER_X_Y)
N1_1=phi2_1*(exp(phi1_1*(x))-1)+phi1_1;
A1_1=(phi1_1*exp(phi2_1*(x))/N1_1)^phi3_1;
B1_1=(exp(phi1_1*(x))-1)/N1_1; P1_1=A1_1*exp(-B1_1*r1_1);

N2_1=phi2_1*(exp(phi1_1*(x-1))-1)+phi1_1;
A2_1=(phi1_1*exp(phi2_1*(x-1))/N2_1)^phi3_1;
B2_1=(exp(phi1_1*(x-1))-1)/N2_1; P2_1=A2_1*exp(-B2_1*r2_1);

N1_2=phi2_2*(exp(phi1_2*(x))-1)+phi1_2;
A1_2=(phi1_2*exp(phi2_2*(x))/N1_2)^phi3_2;
B1_2=(exp(phi1_2*(x))-1)/N1_2; P1_2=A1_2*exp(-B1_2*r1_2);

N2_2=phi2_2*(exp(phi1_2*(x-1))-1)+phi1_2;
A2_2=(phi1_2*exp(phi2_2*(x-1))/N2_2)^phi3_2;
B2_2=(exp(phi1_2*(x-1))-1)/N2_2; P2_2=A2_2*exp(-B2_2*r2_2);

y=(P1_2.*P2_2.*ER_X_Y)./(P1_1.*P2_1.*x);
```

Source file integral2.m:

```
function
y=integral_2(x,phi1_1,phi2_1,phi3_1,phi1_2,phi2_2,
phi3_2,r1_1,r2_1,r1_2,r2_2)
N2_1=phi2_1*(exp(phi1_1*(x-1))-1)+phi1_1;
A2_1=(phi1_1*exp(phi2_1*(x-1))/N2_1)^phi3_1;
B2_1=(exp(phi1_1*(x-1))-1)/N2_1; P2_1=A2_1*exp(-B2_1*r2_1);

N2_2=phi2_2*(exp(phi1_2*(x-1))-1)+phi1_2;
A2_2=(phi1_2*exp(phi2_2*(x-1))/N2_2)^phi3_2;
B2_2=(exp(phi1_2*(x-1))-1)/N2_2; P2_2=A2_2*exp(-B2_2*r2_2);

y=(P2_2.^2./P2_1.^2)*1./x;
```

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