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Methods for Calculating the Sharpe Ratio

Diploma Thesis

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Chapter 1

Introduction

There are several methods for pricing asset characteristics such as the risk-free interest rate, the return of risky assets and Sharpe Ratios. In this thesis the underlying model is always a stochastic growth model. Even when we are restricted to a certain model there are still different approaches for solving for the financial measures, such as the Time Series, the Random Variable and the Stochastic Dynamic Programming approach. The purpose of this thesis is to compare the outcome of those approaches with each other when applied to the same model. In this thesis the time series and the random variable approach are computed but not the stochastic dynamic programming approach. The results for the latter can be found in Grüne and Semmler, Solving Asset Pricing Models with Stochastic Dynamic Programming [7].

Since we will work with a stochastic growth model, which means we work with a model where we do not obtain certain outcomes but usually various possible ones, the concepts of random variables, expected values, etc. are extensively used throughout this paper. Thus chapter 2 will give an introduction to probability and distribution theory. It will also describe the basic features of stochastic growth models as used throughout the following chapters.

Since we want to solve the model for asset price characteristics, we have to close the gap between the consumption paths we find when solving the stochastic growth models and the parameters we need for calculating the financial measures. The intention of chapter 3 is to explain how we get from

the basic pricing equation, which is nothing else but the first order condition for an optimization problem, to measures such as the risk-free interest rate, etc.. We will also find that knowing a consumer's stochastic discount factor is sufficient for the calculation of at least an upper bound for the Sharpe Ratio.

In chapter 4 the reference model as was used in the underlying paper, that is Grüne and Semmler, Solving Asset Pricing Models with Stochastic Dynamic Programming [7], is introduced. This model is consumption based with labour augmenting shocks and given dynamics for the capital development over time. It can be solved analytically. Looking at the capital dynamics in the long run we will discover that independent of starting values the capital will start to oscillate around some sort of "equilibrium point" at an early point in time . Thus one assumes that the financial measures calculated from this model will also converge quickly to some "equilibrium".

Chapter 5, which introduces the time series approach will show, that the assumed convergence behaviour of the Sharpe Ratio cannot be found. In fact, we will find that we have to look very far ahead of the starting point in time before we get at least some kind of "equilibrium". But even then the difference between the outcome of our approach compared to the one of the stochastic dynamic programming approach, is obvious.

In chapter 6 the last approach is introduced. Since we will work with the concept of random variables, all information about the distributions are used. In comparison with the results of the stochastic dynamic programming approach the results we obtain in this context are nearly equal.

Having obtained all results needed to compare the models with each other, and having seen that the outcomes vary rather a lot, chapter 7 will now look for differences in the programmes and approaches. Since the results of the random variable approach are equal to the results of the stochastic dynamic programming approach, this chapter is focussing on explaining the differences between the random variable and time series model. We will start with verifying the time series programm for accurateness. The next section will look for correlations between various parameters of the underlying growth model and will then look in how far the correlations differ from approach to approach. Having figured out that there are differences, we will test the impact

of a change of the standard deviation of our shock variable on the model outcomes.

Reality shows that consumption based models do not fit financial market characteristics to time series data. Therefore asset pricing models with the additional preference of loss aversion were recently proposed by behavioural finance. Thus in chapter 8 we have a closer look at a growth model also including the concept of loss aversion. Consumers are here assumed to become more careful with every loss they experience. In order to fit this concept into our basic stochastic growth model, we have to adjust the model for a factor that reacts on gains or losses the consumer experiences. Since the financial measures have to give considerations to this adjustment, we will re-define the formulas we use for the calculation of the asset price characteristics. In the last section we will compare the results gained from a time series approach to solve that model for Sharpe Ratios with the results obtained by the stochastic dynamic programming as can be found in Grüne and Semmler, Solving Asset Pricing with Loss Aversion [8].

Finally we will summarize and conclude the results found in the previous chapters.

The Appendix will cover the MATLAB codes used for computation and it will give an overview of the included CD.

Chapter 2

Preliminaries

We will begin this chapter with the introduction to probability and distribution theory. These concepts are essential for the different methods we use for solving the underlying model for the asset price characteristics. Furthermore we will define the basic concepts of the stochastic growth model such as the *Marginal Utility of Consumption*.

2.1 Introduction to Probability Theory

Because of the uncertainty about future payoffs that stochastic growth models try to capture we need formal definitions of stochastic variables, i.e. the *Expected Value*, the *Standard deviation*, etc. Therefore it is necessary to give a brief introduction to probability theory as in Greene's Econometric Analysis [6], chp. 3 and 4.

Definition 2.1.1 (Random Variable).

A Random Variable $X : \Omega \rightarrow \mathbb{R}$ is a function from an "event space" Ω to \mathbb{R} . Its realizations or outcomes will be denoted by x . The probability measure is associated with X so that subsets of Ω are assigned a probability $P(\Omega)$. In other words: Until the experiment is performed it is unknown what value X will take. Therefore probabilities are associated with realizations to quantify this uncertainty. The probability that X takes a certain value is denoted as $P(X=x)$.

A random variable is said to be discrete if the set of outcomes is either finite in number or countably infinite. The random variable is continuous if the set of outcomes is uncountable.

Definition 2.1.2 (Probability Distribution).

A probability distribution denoted as $f(x)$ is a listing of values taken by a random variable X and their associated probabilities.

For a discrete random variable,

$$f(x) = P(X = x). \quad (2.1)$$

The axioms of probability require that

1. $0 \leq P(X = x) \leq 1$.
2. $\sum_x f(x) = 1$.

For the continuous case, we can only assign positive probabilities to intervals in the range of x , because the probability associated with any particular point is zero. The *probability density function* (pdf) is defined so that $f(x) \geq 0$ and

1. $P(a \leq x \leq b) = \int_a^b f(x)dx \geq 0$.

The result is the area under $f(x)$ in the interval from a to b . The axioms of probability require that

2. $\int_{-\infty}^{\infty} f(x)dx = 1$.

It is possible that the range of x is not infinite. In that case $f(x) = 0$ is defined anywhere outside the appropriate range.

Definition 2.1.3 (Cumulative Distribution Function).

Let x be a random variable. The probability that x is less than or equal to a is denoted $F(a)$. $F(x)$ is called cumulative distribution function (cdf).

For a discrete random variable,

$$F(x) = \sum_{X \leq x} f(x) = P(X \leq x). \quad (2.2)$$

For a continuous variable,

$$F(x) = \int_{-\infty}^x f(t)dt.$$

For the discrete as well as for the continuous random variable, the cdf has to satisfy the following properties:

1. $0 \leq F(x) \leq 1$.
2. If $x > y$, then $F(x) \geq F(y)$.
3. $F(+\infty) = 1$.
4. $F(-\infty) = 0$.

Definition 2.1.4 (Expected Value).

The mean, or expected value, denoted μ , of a random variable is defined as:

$$E[x] = \begin{cases} \sum_x xf(x) , & \text{if X is discrete,} \\ \int_x xf(x)dx, & \text{if X is continuous.} \end{cases} \quad (2.3)$$

Henceforth the notations \sum_x or \int_x are used for the sum or integral over the entire range of values of x.

Let $g(x)$ be a function of x. The expected value of $g(x)$ is defined as:

$$E[g(x)] = \begin{cases} \sum_x g(x)f(x) , & \text{if X is discrete,} \\ \int_x g(x)f(x)dx, & \text{if X is continuous.} \end{cases} \quad (2.4)$$

Definition 2.1.5 (Variance, Standard Deviation).

The Variance, which must be positive, of a random variable is denoted σ^2 and defined as:

$$Var[x] = E[(x - \mu)^2] \quad (2.5)$$

$$= \begin{cases} \sum_x (x - \mu)^2 f(x) , & \text{if X is discrete,} \\ \int_x (x - \mu)^2 f(x)dx, & \text{if X is continuous.} \end{cases} \quad (2.6)$$

It measures the dispersion of a distribution. Usually, we use σ , the positive square root, to describe a distribution. σ is called the standard deviation of x and can be interpreted as having the same units of measurement as x and μ .

There are some important results that simplify the computation of the expected value and the variance.

Theorem 2.1.6. *Let X be a random variable.*

1. The computation of the variance is simplified by using the following equation:

$$\text{Var}[X] = E[X^2] - \mu^2 \quad (2.7)$$

2. For constants a and b :

$$E[a + bX] = a + bE[X], \quad (2.8)$$

which implies, for any constant a , that $E[a] = a$.

3. For constants a and b :

$$\text{Var}[a + bX] = b^2\text{Var}[X], \quad (2.9)$$

which implies, for any constant a , that $\text{Var}[a] = 0$.

Proof: Theorem 2.1.5.

Consider the discrete case, the continuous one is analogous.

- 1.

$$\begin{aligned} \text{Var}[x] &= \sum_x (x - \mu)^2 f(x) \\ &= \sum_x (x^2 - 2x\mu + \mu^2) \\ &= \underbrace{\sum_x x^2 f(x)}_{=E[x^2]} - 2\mu \underbrace{\sum_x x f(x)}_{=\mu} + \mu^2 \underbrace{\sum_x f(x)}_{=1} \\ &= E[x^2] - 2\mu^2 + \mu^2 = E[x^2] - \mu^2 \end{aligned}$$

2.

$$\begin{aligned}
E[a + bx] &= \sum_x (a + bx)f(x) \\
&= a \underbrace{\sum_x f(x)}_{=1} + b \underbrace{\sum_x xf(x)}_{=E[x]} \\
&= a + bE[x]
\end{aligned}$$

3.

$$\begin{aligned}
Var[a + bx] &= \underbrace{E[(a + bx)^2]}_{=E[a^2 + 2abx + b^2x^2]} - \underbrace{E[a + bx]^2}_{=(a + bE[x])^2} \\
&= a^2 + 2abE[x] + b^2E[x^2] - a^2 - 2abE[x] - b^2E[x]^2 \\
&= b^2(E[x^2] - E[x]^2) \\
&= b^2Var[x]
\end{aligned}$$

□

In our stochastic growths model we will work with two random variables. It is therefore useful to introduce the *joint density function*, the *conditional mean* and the *conditional variance*.

Let X and Y be two random variables. The joint density function, $f(x, y)$, satisfies:

$$P(a \leq x \leq b, c \leq y \leq d) = \begin{cases} \sum_{a \leq x \leq b} \sum_{c \leq y \leq d} f(x, y), & \text{discrete case,} \\ \int_a^b \int_c^d f(x, y) dx dy & \text{continuous case.} \end{cases} \quad (2.10)$$

The requirements that go along with the requirements in the univariate case, are:

$$\begin{aligned}
f(x, y) &\geq 0 & , \\
\sum_x \sum_y f(x, y) &= 1 & \text{, if X and Y are discrete,} \\
\int_x \int_y f(x, y) dx dy &= 1, & \text{if X and Y are continuous.}
\end{aligned} \quad (2.11)$$

We need the definition for the marginal probability distribution in order to define expectations in a joint distribution.

Definition 2.1.7 (Marginal Probability Density).

The marginal probability density is also called marginal probability distribution. It is defined with respect to an individual variable and obtained from the joint distribution by summing or integrating out the other variable.

$$f_x(x) = \begin{cases} \sum_y f(x, y), & \text{in the discrete case,} \\ \int_y f(x, s) ds, & \text{in the continuous case.} \end{cases} \quad (2.12)$$

Analogous for $f_y(y)$.

Definition 2.1.8 (Expected Values and Variances in a Joint Distribution).

Let X and Y be random variables. Means and Variances of the variables are calculated with respect to the marginal distributions. For the mean of x in a discrete distribution,

$$E[x] = \sum_x x f_x(x) \quad (2.13)$$

$$= \sum_x x \left[\sum_y f(x, y) \right] \quad (2.14)$$

$$= \sum_x \sum_y x f(x, y) \quad (2.15)$$

The means of a variable in the continuous case are defined likewise, using integration instead of summation.

The computation of variances works in the same manner:

$$Var[x] = \sum_x (x - E[x])^2 f(x) \quad (2.16)$$

$$= \sum_x \sum_y (x - E[x])^2 f(x, y). \quad (2.17)$$

Similarly for the continuous case, again using integration.

Definition 2.1.9 (Covariance and Correlation).

Let X and Y be random variables. For any function $g(x, y)$,

$$E[g(x, y)] = \begin{cases} \sum_x \sum_y g(x, y) f(x, y) & , \text{ in the discrete case} \\ \int_x \int_y g(x, y) f(x, y) dy dx, & \text{ in the continuous case.} \end{cases} \quad (2.18)$$

The covariance of x and y is:

$$Cov[x, y] = E[(x - \mu_x)(y - \mu_y)] \quad (2.19)$$

$$= E[xy] - \mu_x \mu_y \quad (2.20)$$

$$= \sigma_{xy}. \quad (2.21)$$

For the analysis the sign of the covariance is important, because it will indicate the direction of the covariation of X and Y . Consider an investment strategy that covaries positively with consumption, i.e. if consumption goes up, so does the revenue of the investment strategy. This sort of investment would not save from loss if income decreases and, as a result of it, consumption decreases.

Since the magnitude of the covariation depends on the scales of measurement, the preferable measure is the correlation coefficient.

Definition 2.1.10 (Correlation Coefficient).

The correlation coefficient has the same sign as the covariance but is unaffected by any scaling of the variables. Hence its value is always between -1 and 1 .

$$r[x, y] = \rho_{xy} \quad (2.22)$$

$$= \frac{\sigma_{xy}}{\sigma_x \sigma_y}, \quad (2.23)$$

where σ_x and σ_y are the standard deviations of x and y , respectively.

In econometric modelling, conditioning and the use of conditional distributions play a pivotal role. Again, we will just consider the bivariate case.

Definition 2.1.11 (Conditional Distribution).

There is a conditional distribution over y for each value of x . The conditional

densities are:

$$f(y | x) = \frac{f(x, y)}{f_x(x)} \quad (2.24)$$

and

$$f(x | y) = \frac{f(x, y)}{f_y(y)}. \quad (2.25)$$

Definition 2.1.12 (Conditional Mean and Conditional Variance).

The conditional mean is the mean of the conditional distribution:

$$E[y | x] = \begin{cases} \sum_y y f(y | x), & \text{if } y \text{ is discrete} \\ \int_y y f(y | x) \, dy, & \text{if } y \text{ is continuous.} \end{cases} \quad (2.26)$$

The conditional mean is also called the *regression* of y on x .

Similar to the mean, the conditional variance is the variance of the conditional distribution:

$$\text{Var}[y | x] = E[(y - E[y | x])^2 | x] \quad (2.27)$$

$$= \int_y (y - E[y | x])^2 f(y | x) dy, \text{ if } y \text{ is continuous} \quad (2.28)$$

$$\text{or} \quad (2.29)$$

$$\text{Var}[y | x] = \sum_y (y - E[y | x])^2 f(y | x), \text{ if } y \text{ is discrete.} \quad (2.30)$$

Again, the computation can be simplified by:

$$\text{Var}[y | x] = E[y^2 | x] - (E[y | x])^2. \quad (2.31)$$

Definition 2.1.13 (σ -Algebra).

Let Ω be a set. Then a σ -algebra \mathcal{F} is a nonempty collection of subsets of Ω such that the following hold:

1. $\Omega \in \mathcal{F}$
2. if $A \in \mathcal{F}$ then $A^c \in \mathcal{F}$, where A^c denotes the complement of A ,

3. if $(A_n)_n$ is a sequence of elements of \mathcal{F} , then $\bigcup_n A_n \in \mathcal{F}$.

If S is any collection of subsets of \mathcal{F} , then we can always find a σ -algebra containing S , namely the power set of Ω , $\mathcal{P}(\Omega)$. By taking the intersection of all σ -algebras containing S , we obtain the smallest such σ -algebra. We call the smallest σ -algebra containing S , the σ -algebra generated by S .

Definition 2.1.14 (Borel sets).

The Borel σ -algebra is defined to be the smallest σ -algebra generated by the open sets in Ω .

Definition 2.1.15 (Independent and Identically distributed Random Variables).

Let X and Y be random variables.

$$X \text{ and } Y \text{ are independent} \iff f(x, y) = f_x(x)f_y(y) \quad (2.32)$$

Remark 2.1.16. If two random variables X and Y are independent then their covariance is zero, that means they are uncorrelated. And for any two functions $g_1(x)$ and $g_2(y)$ then

$$E[g_1(x)g_2(y)] = E[g_1(x)]E[g_2(y)]. \quad (2.33)$$

In order to look at the convergence behaviour of a sequence of random variables x_n , we will introduce the concept of convergence in distribution:

Definition 2.1.17 (Convergence in Distribution). Assume x_n to be a sequence of random variables and let x_n have cdf $F_n(x)$. x_n converges in distribution to a random variable x if

$$\lim_{n \rightarrow \infty} |F_n(x) - F(x)| = 0$$

at all continuity points of $F(x)$. $F(x)$ is the cdf of x .

The basic form of the most important theorem in econometrics which helps to describe the statistical properties of estimators when their exact distributions are unknown, is given in the following theorem.

Theorem 2.1.18 (Central Limit Theorem).

Let x_1, \dots, x_n be a random sample from a probability distribution with mean μ and finite variance σ^2 . Then for $\bar{x}_n = \frac{1}{n} \sum_{i=1}^n x_i$ the *Lindbergh-Levy* variant for the mean of a univariate distribution holds. Formally:

$$\sqrt{n}(\bar{x}_n - \mu) \xrightarrow{d} N[0, \sigma^2], \quad (2.34)$$

where $N[0, \sigma^2]$ is the the normal distribution with mean 0 and standard deviation σ . \xrightarrow{d} stands for the limiting distribution and means that if x_n converges in distribution to x , and $F(x)$ is the cdf of x , then $F(x)$ is the limiting distribution of x_n .

Thus $\sqrt{n}(\bar{x}_n - \mu) \xrightarrow{d} N[0, \sigma^2]$ can also be thought of representing the fact that the relationship is an asymptotic one, i.e. with n increasing without boundary the distribution $\sqrt{n}(\bar{x}_n - \mu)$ approaches the normal distribution of with mean zero and variance σ^2 . See also A.W. Lo, [11], p.38..

2.2 Stochastic Growth Model

According to Mirman and Brock [2], we will introduce the basic stochastic growth model.

One tradition of asset pricing models is based on the stochastic growth model with production. In these models it is crucial how consumption is modelled. The goal is to find an optimal consumption path in order to deduce the values of investment strategies. The randomness is assumed to occur in the production function of firms as labor augmenting technical progress. As a result consumption and dividends can be assumed to be endogenous. In dealing in this uncertain context, we will maximize the expected sum of discounted utilities.

Definition 2.2.1 (Utility of Consumption).

Utility of Consumption, denoted $u(c)$, is the welfare an individual gets by consuming a certain amount of goods. We assume consumers to spend their available income so as to maximize their *utility function*, $u(\cdot)$.

Definition 2.2.2 (Utility Function).

The utility function $u(\cdot)$ is the expression economists use to show utility as

a function of an individual's consumption. We assume the utility, $u(c)$, is increasing in the quantity of each good consumed. We require the utility function to fulfill the usual properties, that is:

$$u' > 0, \quad u'' < 0.$$

The second property denotes concavity, which is important to express a consumer's declining marginal value of additional consumption. To guarantee any consumption at all, we also require the utility function to satisfy $u'(0) = +\infty$.

From now on time subscripts will be used only when necessary to explicitly distinguish events in different periods.

Consider a one sector growth model of an economy in discrete time with future production uncertainties. The production function for this economy is given by:

$$Y_t = F(K_t, L_t; \epsilon_t), \tag{2.35}$$

where Y_t , K_t and L_t are output, capital and labor at time t , respectively. ϵ_t is the random variable representing uncertainty. As usual, we assume that our production function has constant returns to scale in its first two arguments. That means, that multiplying both arguments by any nonnegative constant α causes output to change by the same factor.

Formally, but without the time coefficient:

$$F(\alpha L, \alpha K; \epsilon) = \alpha F(L, K; \epsilon), \quad \text{for all } \alpha \geq 0. \tag{2.36}$$

Because of the production function's homogeneity of degree 1, we can work with the production function in the following form. Let $\alpha = \frac{1}{L}$, we yield:

$$y_t \equiv \frac{Y_t}{L_t} = F\left(1, \frac{K_t}{L_t}; \epsilon_t\right) = f(k_t, \epsilon_t). \tag{2.37}$$

Here k_t is the economy's capital stock per unit of labor and y_t is output per unit of labor. k_t is also called capital labor ratio.

Assumption 2.2.3 (Assumptions on f).

Let ρ be the possible values of the random variable ϵ_t . $f(\cdot, \rho)$ is endowed

with the common properties of production functions. Thus we assume:

$$\begin{aligned} f(0, \rho) &= 0, \\ f'(\cdot, \rho) &> 0, \\ f''(\cdot, \rho) &< 0, \end{aligned}$$

for all values of ρ . In addition, $f(\cdot, \rho)$ is assumed to satisfy the *Inada* conditions, namely:

$$\begin{aligned} f'(0, \rho) &= +\infty, \\ f'(\infty, \rho) &= 0, \end{aligned}$$

for all values of ρ . Finally f and f' are continuous in k and ϵ jointly.

Now, take a closer look at the statistics of the random variable ϵ_t . Since the probability space $(\Omega_t, \mathcal{F}_t, P_t)$ is assumed to be the same in each period, time subscripts will be dropped for convenience and the state space will be denoted by (Ω, \mathcal{F}, P) . Here, let Ω be the "event space", the arbitrary, nonempty set of possible "states of the world" that influence the production function. $\omega \in \Omega$ represents the occurrence of an exogenous event that effects the production function. To be mathematically rigorous, let \mathcal{F} be a collection of subsets of Ω . The set \mathcal{F} is a σ -algebra of subsets of Ω . P is supposed to be a probability measure, i.e. P is a nonnegative σ -additive function on \mathcal{F} . That means $P(\Omega) = 1$, $P(F) = \sum_{i=1}^m P(F_i) \forall F \in \mathcal{F}$ and every finite partition $\{F_i, i = 1, \dots, m\}$ of F in \mathcal{F} and $\forall F \in \mathcal{F}$ is $\{F_i, i = 1, 2, \dots\}$, $F_i \neq F_j \forall i \neq j$, $\cup_{i=1}^{\infty} F_i = F$ and $\cup_{i=1}^{\infty} P(F_i) = P(F)$.

Thus the probability that the state of the world ω is an element of F is given by

$$P(F) = P[\omega \in F]. \quad (2.38)$$

In order to translate random happenings in measurable values the random variable ϵ_t is introduced that takes our measure space (Ω, \mathcal{F}, P) into the real line $(\mathbb{R}, \mathcal{B})$, where \mathcal{B} is the set of Borel sets defined as above.

It is important to note that the random variables ϵ_t are assumed to be independent and identically distributed. This assumption is strongly simplifying whereas the assumption about the time independency of the measure space remains crucial to the limit theorem.

$\epsilon : \Omega \rightarrow \mathbb{R}$ generates a measure, $\nu(\cdot)$, on the Borel subsets of the real line.

$$\nu(z \in S) = P\{\omega \in \epsilon^{-1}(S)\},$$

where

$$\epsilon^{-1}(S) \in \mathcal{F} \text{ for all } S \in \mathcal{B},$$

and

$$\epsilon^{-1}(S) = \{\omega : z(\omega) \in S\}.$$

Thus the numerical random value z with associated measure $\nu(\cdot)$ represents the statistics of the random "states of the word".

Output is either increased or decreased for all values of k by the random event ϵ . Moreover it is assumed that the random events are indexed in a way as to have

$$\frac{\partial f(k, \epsilon)}{\partial \epsilon} > 0, \quad \text{for all } k. \quad (2.39)$$

Furthermore, it is assumed that the random event can only affect production in a "compact" way. In other words, numbers $0 < \alpha < \beta < \infty$ exist such that

$$\epsilon \in [\alpha, \beta]$$

and for all $x > 0$,

$$\infty > f(k, \beta) > f(k, \alpha) > 0. \quad (2.40)$$

Without loss of generality it is assumed that for every $\rho > \alpha$, $\nu([\alpha, \rho]) > 0$. Since, if not, $\alpha = \sup\{\eta : \nu([0, \eta]) = 0\}$ could be chosen. Accordingly, $\nu([\rho, \beta]) > 0$ for each $\rho < \beta$.

Now, consider the growth process which is represented by the following difference equation

$$c_t + k_t = f(k_{t-1}, \epsilon_{t-1}), \quad (2.41)$$

where c_t is an optimal consumption policy in the sense that consumption is chosen so as to maximize the expected sum of discounted utilities. We will also assume an infinite time horizon.

Definition 2.2.4 (Maximization Problem).

In this set-up the maximization problem becomes:

$$\max E_t \left[\sum_{t=0}^{\infty} \delta_t u(c_t) \right] \quad (2.42)$$

subject to the constraints

$$\begin{aligned} c_t + k_t &= f(k_{t-1}, \epsilon_{t-1}), \quad t = 1, 2, \dots \\ c_0 + k_0 &= s, \quad k_0, c_0 \geq 0, \end{aligned}$$

where $s > 0$ is the given initial data of the problem representing the historically given capital stock. δ is the subjective discount rate and it holds: $\delta \in (0, 1)$.

Consider the definition of β and the conditions that hold for the production function. There exists a k_β so that for all $k > k_\beta$

$$f(k, \beta) < k.$$

Thus it is not possible to sustain capital independent from the state of the world if $k > k_\beta$.

Chapter 3

Stochastic Growth Model and Financial Measures

In the context of this paper we have to compute every financial measure with respect to the set-up of a stochastic growth model. In order to get our formulas in the necessary way we look without loss of generality at a discrete, two period decision problem, cf Cochrane [4] p.6 - p.19. We will see that every infinite decision problem is equivalent to the two-period problem. It is also always possible to look at the constraints to such a problem from the perspective of consumption.

Let us start with the two-period problem and derive the basic pricing equation. For every maximization problem of the form:

$$\begin{aligned} & \max_{\xi} u(c_t) + E_t[\beta u(c_{t+1})] \\ & \text{subject to} \\ & c_t = e_t - p_t \xi \\ & c_{t+1} = e_{t+1} + x_{t+1} \xi, \end{aligned}$$

where ξ denotes the fraction of income that is invested, e_t is the income of period t and x_t is period t 's payoff, the *first order condition* for an optimal

consumption and portfolio choice is:

$$\frac{d}{d\xi}[u(e_t + p_t\xi) + E_t[\beta u(e_{t+1} + x_{t+1}\xi)]] = 0 \quad (3.1)$$

$$\iff p_t = E_t[\underbrace{\beta \frac{u'(c_{t+1})}{u'(c_t)}}_{=m_{t+1}} x_{t+1}] = E_t[m_{t+1}x_{t+1}]. \quad (3.2)$$

Usually, for stocks, the one-period payoff is the next price and the dividend, i.e. $x_{t+1} = p_{t+1} + d_{t+1}$.

Now, consider an infinite-period problem, where the investor can choose a dividend stream d_{t+1} over time and purchase it at time t for a price p_t . The maximization problem becomes:

$$\max_{\xi} E_t \left[\sum_{j=0}^{\infty} \beta^j u(c_{t+j}) \right]$$

subject to:

$$\begin{aligned} c_t &= e_t - p_t \xi \\ c_{t+j} &= e_{t+j} + \xi d_{t+j} \end{aligned}$$

As in the two-period case the First Order condition gives us the basic pricing equation directly.

$$\begin{aligned} p_t &= E_t \left[\sum_{j=1}^{\infty} \beta^j \frac{u'(c_{t+j})}{u'(c_t)} d_{t+j} \right] \\ &= E_t \left[\beta \frac{u'(c_{t+1})}{u'(c_t)} d_{t+1} + \sum_{j=2}^{\infty} \beta^j \frac{u'(c_{t+j})}{u'(c_t)} d_{t+j} \right] \\ &= E_t \left[m_{t+1} d_{t+1} + \beta \frac{u'(c_{t+1})}{u'(c_t)} \sum_{j=2}^{\infty} \beta^{-1} \frac{u'(c_t)}{u'(c_{t+1})} \beta^j \frac{u'(c_{t+j})}{u'(c_t)} d_{t+j} \right] \\ &= E_t \left[m_{t+1} d_{t+1} + m_{t+1} \sum_{j=2}^{\infty} \beta^{j-1} \frac{u'(c_{t+j})}{u'(c_{t+1})} d_{t+j} \right] \\ &= E_t \left[m_{t+1} (d_{t+1} + \sum_{j=1}^{\infty} \beta^j \frac{u'(c_{t+1+j})}{u'(c_{t+1})} d_{t+1+j}) \right] \\ &= E_t [m_{t+1} (d_{t+1} + p_{t+1})] \\ &= E_t [m_{t+1} x_{t+1}] \end{aligned}$$

Starting from the basic pricing equation we can redefine the asset pricing characteristics.

R^f is still the risk-free interest rate. Its value is known in advance and every return is how many units of consumption, e.g. dollars, you get tomorrow, if you pay one unit today. Thus the basic pricing equation for the risk-free interest rate becomes $1 = E[mR^f] = E[m]R^f$. Thus

$$R^f = \frac{1}{E[m]}. \quad (3.3)$$

As usual the time subscripts are dropped for reasons of convenience. Using the latter equation and the fact that the covariance can be computed as $Cov[m, x] = E[mx] - E[m]E[x]$ we get:

$$E[R] - R^f = -R^f Cov[m, R] \quad (3.4)$$

Proof: of Equation (3.4).

$$\begin{aligned} p &= E[mx] \\ \iff p &= Cov[m, x] + \underbrace{E[m]}_{\frac{1}{R^f}} E[x] \\ \iff p &= \underbrace{Cov[m, x]}_{\text{risk adjustment}} + \underbrace{\frac{1}{R^f} E[x]}_{\text{asset's price in a risk neutral world}} \end{aligned}$$

Now consider

$$\begin{aligned} 1 = E[mR] &\iff 1 = Cov(mR) + \frac{1}{R^f} E[R] \\ &\iff E[R] - R^f = -R^f Cov[m, R] \end{aligned}$$

□

One important index for our analysis is the Sharpe Ratio (SR). It measures the risk-adjusted performance of an investment asset. The formal definition is as follows:

Definition 3.0.5 (Sharpe Ratio).

The Sharpe Ratio is the ratio of the excess expected return, $\mu - R_f$, of an investment to its return volatility or standard deviation, σ .

$$SR \equiv \frac{\mu - R_f}{\sigma} \quad (3.5)$$

where μ denotes the expected return of the investment asset and R_f the return of another "standard" asset, mostly referred to as the return of a risk-free asset, i.e. a bond.

The intention of the SR is to measure the surplus return per unit of risk. By comparison of two assets usually the one with the higher positive Sharpe Ratio is chosen as investment strategy.

Thus we can express the Sharpe Ratio in the following way:

$$SR = \left| \frac{E[R] - R^f}{\sigma_R} \right| \quad (3.6)$$

$$= \frac{-R^f Cov[m, R]}{\sigma_R}, \quad (3.7)$$

where

$$R_{t+1} = \frac{d_{t+1} + p_{t+1}}{p_t} \quad (3.8)$$

is the gross return and d_t denotes the dividend on the asset in period t .

We also have the opportunity to calculate an upper bound for the Sharpe Ratios.

Theorem 3.0.6. The upper bound, SB, for the Sharpe Ratio is given by:

$$SB = \frac{\sigma(m)}{E[m]} \quad (3.9)$$

Proof: of Theorem (3.0.7.)

$$\begin{aligned}
SR &= \frac{-R^f \text{Cov}[m, R]}{\sigma_R} = \frac{-\frac{1}{E[m]} \text{Cov}[m, R]}{\sigma_R} \\
&= -\frac{\sigma_m}{E[m]} \underbrace{\frac{\text{Cov}[m, R]}{\sigma_m \sigma_R}}_{=\rho_{m,R}} = \frac{\sigma_m}{E[m]} \underbrace{-\rho_{m,R}}_{-1 \leq \rho_{m,R} \leq 1} \\
&\leq \frac{\sigma_m}{E[m]} = SB
\end{aligned}$$

□

Chapter 4

Reference Model

We will now have a closer look at the most basic stochastic growth model, as in Grüne and Semmler [7] in chapter 5. The general set-up as in Mirman and Brock [2] was already introduced in chapter 2.

4.1 The Model

In our model we will now have log utility, i.e. $u(c_t) = \ln u(c_t)$, aggregate capital stock and labour augmenting technology shocks. In detail, the problem set-up is as follows:

$$V(k, z) := \max_{c_t} E\left[\sum_{t=0}^{\infty} \beta^t \ln(c_t)\right] \quad (4.1)$$

subject to

$$\left. \begin{array}{l} k_{t+1} = z_t A k_t^\alpha - c_t \\ \ln z_{t+1} = \rho \ln z_t + \epsilon_t \end{array} \right\} =: \varphi((k, z), c, \epsilon)$$

A , α and ρ are real constants and the ϵ_t are i.i.d. random variables with zero mean. For numerical reasons we substitute $\ln z_t$ by y_t . Important for the time series is the fact that ϵ_t is chosen as a Gaussian distributed random variable with standard deviation $\sigma = 0.008$. The domain in which we will compute the

model, $\Omega = [0.1, 10] \times [-0.32, 0.32]$, can be controlled invariant, that means there exists an optimal consumption path c such that $\varphi((k, z), c, \epsilon) \in \Omega$ for any $(k, z) \in \Omega$ and for all ϵ . The difference between the model in Grüne and Semmler, [7], p. 15 and this model is, that, since we do not work on grids, we do not have to restrict ϵ on the interval $[-0.032, 0.032]$, which is very handy for the time series approach. The topic restriction will be discussed again in chapter 6, when the random variable approach is introduced.

This model is equivalent to the stochastic growth model of chapter 2. the difference is just, that in our model the discount factor is given by β instead of δ , and we will give initial values for z and k instead an initial value for $s = c_0 + k_0$ as was in the Mirman and Brock,[2] set-up.

4.2 The Optimal Value and Control Function

An interesting feature of the value function (4.1) is that it can also be written as

$$V(k, z) = \max_c E[u(c(k, z)) + \beta V(\varphi((k, z), c(k, z), \epsilon))] \quad (4.2)$$

This is the *Bellman equation*. It reduces the multi-period optimization to a two-stage problem, effectively. At first we maximize current utility by taking into account that the choice c_t affects future possibilities through k_{t+1} . Then, in the future we also maximize so that our outcome can be summarized by the expected discounted future value function. This insight in dynamic programming is referred to as the *optimality principle*, see Geraats, [5]

As mentioned above our model can be solved analytically for the policy function in feedback form. Thus a sequence of optimal consumption can be derived and used for the numerical solution of the asset's price and the Sharpe Ratio.

The following remark will give the optimal value and control function, see Grüne, [9].

Remark 4.2.1. $V^*(k, z)$ is the optimal value function for problem (4.1) given by

$$V^*(k, z) = B + C \ln(k) + D \ln(z)$$

with

$$B = \frac{\ln(1 - \alpha\beta)A + \frac{\alpha\beta}{1 - \alpha\beta} \ln \alpha\beta A}{1 - \beta},$$

$$C = \frac{\alpha}{1 - \alpha\beta}$$

and

$$D = \frac{1}{(1 - \alpha\beta)(1 - \rho\beta)}$$

Then, the optimal consumption sequence c for the stochastic growth model with log utility and the above dynamics is given by:

$$c^*(k, z) = (1 - \alpha\beta)Az_t k_t^\alpha \quad (4.3)$$

Proof: of Remark 4.2.1.

We will start by calculating the optimal control function to the given optimal value function. After that we will verify that the optimal value function satisfies the optimality principle, which is for all k_t, z_t holds:

$$V(k_t, z_t) = \max_{c_t} E_t[u(c_t(k_t, z_t)) + \beta V(\varphi((k_t, z_t), c_t(k_t, z_t), \epsilon_t))]$$

1. In order to find the optimal control function, we will look at the necessary or first order condition (FOC) and the sufficient or second order condition (SOC) for a maximum.

(a) The necessary condition for a maximum is given by: $V'(k_t, z_t) = 0$.

$$\begin{aligned} V'(k_t, z_t) &= \frac{\partial E_t[u(c_t(k_t, z_t)) + \beta V(\varphi((k_t, z_t), c_t(k_t, z_t), \epsilon_t))]}{\partial c_t} \\ &= \frac{\partial E_t[\ln c_t + \beta(B + C \ln Az_t k_t^\alpha - c_t + D \ln \rho z_t + \epsilon_t)]}{\partial c_t} \\ &= \frac{1}{c_t} + \beta \left(-C \frac{1}{Az_t k_t^\alpha - c_t} \right) \\ &\stackrel{FOC}{=} 0. \end{aligned}$$

$$\begin{aligned} \Rightarrow \quad & \frac{Az_t k_t^\alpha}{c_t} - 1 - \beta C = 0 \\ & \iff \frac{Az_t k_t^\alpha}{1 + \beta C} = c_t \end{aligned}$$

$$\text{with } C = \frac{\alpha}{1 - \alpha\beta}$$

$$\Rightarrow (1 - \alpha\beta)Az_t k_t^\alpha = c_t^*.$$

- (b) If c_t^* maximizes V^* and V^* is twice differentiable, then $V^{*''}(k_t, z_t; c_t^*) \leq 0$ has to hold. (SOC)

$$\begin{aligned} V''(k_t, z_t) &= \frac{\partial^2 E_t[u(c_t(k_t, z_t)) + \beta V(\varphi((k_t, z_t), c_t(k_t, z_t), \epsilon_t))]}{\partial c_t^2} \\ &= E_t\left[-\frac{1}{c_t^2} - \frac{-\beta C(-1)}{(Az_t k_t^\alpha - c_t)^2}\right] \end{aligned}$$

with c_t^*

$$\begin{aligned} \Rightarrow \quad & E_t\left[-\underbrace{\frac{1}{((1 - \alpha\beta)Az_t k_t^\alpha)^2}}_{>0} - \underbrace{\frac{\beta C}{(\alpha\beta Az_t k_t^\alpha)^2}}_{>0}\right] \\ & < 0. \end{aligned}$$

2. c_t^* is the optimal control function, which means that c_t^* maximizes $V^*(k_t, z_t)$ at every point in time and for all (k_t, z_t) .

$V^*(k_t, z_t) = \max_{c_t} E_t[u(c_t(k_t, z_t)) + \beta V^*(\varphi((k_t, z_t), c_t(k_t, z_t), \epsilon_t))]$ has to hold if V^* is the optimal value function.

Thus it is sufficient to show that $V^*(k_t, z_t)$ satisfies

$V^*(k_t, z_t) = E_t[u(c_t^*(k_t, z_t)) + \beta V^*(\varphi((k_t, z_t), c_t^*(k_t, z_t), \epsilon_t))]$ in order to prove that it is the optimal value function.

Let us now start by looking at the right side of the above formula and substituting $\ln z_t$ by y_t :

$$\begin{aligned} & E_t[V^*(\varphi((k_t, y_t), c_t^*(k_t, y_t), \epsilon_t))] \\ &= E_t\left[B + C \underbrace{\ln(e^{y_t} A k_t^\alpha - (1 - \alpha\beta) A e^{y_t} k_t^\alpha)}_{\ln(\alpha\beta A) + y_t + \alpha \ln(k_t)} + D(\rho y_t + \epsilon_t)\right] \end{aligned}$$

$$\begin{aligned}
&= B + C (\ln(\alpha\beta A) + y_t + \alpha \ln(k_t)) + D \rho y_t + D \underbrace{E_t[\epsilon_t]}_{=0} \\
&\stackrel{B,C,D}{=} \frac{\ln((1-\alpha\beta)A) + \frac{\alpha}{1-\alpha\beta} \ln(\alpha\beta A)}{1-\beta} + \frac{\alpha(1-\rho\beta)+\rho}{(1-\alpha\beta)(1-\rho\beta)} y_t \\
&\quad + \frac{\alpha^2}{1-\alpha\beta} \ln(k_t) \\
\Rightarrow V^*(k_t, z_t) &= E_t[u(c_t(k_t, z_t)) + \beta V^*(\varphi((k_t, z_t), c_t^*(k_t, z_t), \epsilon_t))] \\
&= E_t[\ln((1-\alpha\beta)e_t^y A k_t^\alpha) + \beta V^*(\varphi((k_t, z_t), c_t^*(k_t, z_t), \epsilon_t))] \\
&= \ln((1-\alpha\beta)A) y_t \alpha \ln(k_t) + \beta E_t[V^*(\varphi((k_t, z_t), c_t^*(k_t, z_t), \epsilon_t))] \\
&= \ln((1-\alpha\beta)A) y_t \alpha \ln(k_t) \\
&\quad + \beta \frac{\ln((1-\alpha\beta)A) + \frac{\alpha}{1-\alpha\beta} \ln(\alpha\beta A)}{1-\beta} \\
&\quad + \frac{\alpha(1-\rho\beta)+\rho}{(1-\alpha\beta)(1-\rho\beta)} y_t + \frac{\alpha^2}{1-\alpha\beta} \ln(k_t) \\
&= \underbrace{\frac{\ln((1-\alpha\beta)A) + \frac{\alpha\beta}{1-\alpha\beta} \ln(\alpha\beta A)}{1-\beta}}_{=B} \\
&\quad + \underbrace{\frac{(1-\alpha\beta + \alpha\beta)(1-\rho\beta) + \rho\beta}{(1-\alpha\beta)(1-\rho\beta)}}_{=D} y_t \\
&\quad + \underbrace{\frac{\alpha}{1-\alpha\beta}}_C \ln(k_t) \\
&= B + C \ln(k_t) + D y_t
\end{aligned}$$

□

As we have seen in the last chapter $m(c_t, c_{t+1}) = \beta \frac{u'(c_{t+1})}{u'(c_t)}$. For our optimal control function c , m is given by:

$$m(c_t, c_{t+1}) = \beta \frac{c_t}{c_{t+1}} \quad (4.4)$$

The last thing we need before we can start calculating the Sharpe Ratios is the price formula.

Remark 4.2.2. The price for a wealth portfolio with log utility is propor-

tional to consumption. In our case the formula is given by:

$$p_t = \frac{\beta}{1 - \beta} (1 - \alpha\beta) A z_t k_t^\alpha \quad (4.5)$$

Proof: of Remark 4.2.2.

We need to look at the present-value formula for the price of a wealth portfolio. In this context the wealth portfolio is a claim to all future consumption. We yield:

$$\begin{aligned} p_t &= E_t \left[\sum_{j=1}^{\infty} \beta^j \frac{u'(c_{t+j})}{u'(c_t)} c_{t+j} \right] \\ &= E_t \left[\sum_{j=1}^{\infty} \beta^j \frac{c_t}{c_{t+j}} c_{t+j} \right] \\ &\stackrel{\text{geometric series}}{=} \frac{\beta}{1 - \beta} c_t \end{aligned}$$

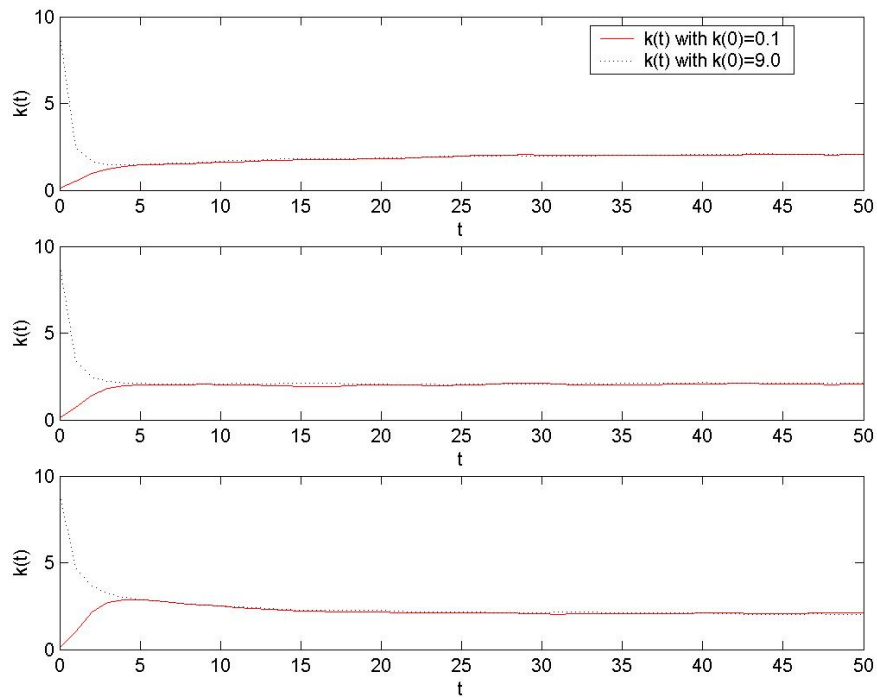
□

We will need these formulas for both approaches to a solution of the maximization problem.

4.3 The Dynamics in the Long Run

Figure 4.1 will give a rough impression of paths k to different start values of k and z and along optimal trajectories. It can easily be seen that all trajectories oscillate around the point $(k, \ln z) \approx (2, 0)$ in the long run.

In the next two chapters we will work through solutions of the problem of calculating the Sharpe Ratio by looking at the time series of the dynamics and as the second approach we will look at $m(x_t)$ as a random variable. In chapter 7 we will compare both attempts with each other and with the analytical solution as can be found in Grüne and Semmler, [7].

Figure 4.1: k -components of optimal trajectories for $z(0) = \{-0.32, 0, 0.32\}$

Chapter 5

Time Series Approach

In this chapter we will solve the problem of calculating the Sharpe Ratio as described in the last chapter by modelling the time series of our dynamics, z and k . Both need to be modelled in order to find possible paths for consumption c . As we have already seen, all time series for the capital stock, k , independent of their initial values oscillate around some kind of "equilibrium" point. Because of that and the fact, that we are just interested in the long time behaviour of our economy, it is enough to always model one path instead of several ones as it is usually done in Monte Carlo Simulations.

5.1 Random Number Generators

In our stochastic growth model the stochastic factor, ϵ , is said to be standard normally distributed with zero mean and a standard deviation of 0.008. Therefore we will need a standard normally distributed random variable. Since every simulation of random variables is a deterministic operation the numbers we generate are said to be *pseudo random variables*. Since there is no Pseudo Random Number Generator with that distribution implemented under MATLAB, it is necessary to implement one. A way to do so was described in "Finanzderivate mit MATLAB" by M. Günther and A. Jüngel, [10].

The first step is to produce uniformly distributed pseudo random variables

in the interval $[0, 1]$,

$$Y \sim U[0, 1],$$

which are transformed into normally distributed random numbers by using a function h .

$$Z := h(Y) \sim N(0, 1)$$

Both terms were already defined in chapter 1. The exact implementation of the algorithms used in this chapter are in the appendix.

Step 1 The uniformly distributed random numbers are produced by lagged Fibonacci-Generators based on Tausworth.

$$\begin{aligned} & \text{for } i \geq \max \mu, \nu \\ & X_i := (X_{i-\mu} + X_{i-\nu}) \pmod{M}, \\ & U_i := X_i/M, \end{aligned}$$

where $X_1, \dots, X_{\mu, \nu}$ are generated by a standard MATLAB random number generator.

Step 2 Uniformly distributed random numbers are transformed by the algorithm based on Box-Muller: Generate $U_1, U_2 \sim U[0, 1]$. Then $Z_1 = \sqrt{-2 \ln U_1} \cos(2\pi U_2)$ and $Z_2 = \sqrt{-2 \ln U_1} \sin(2\pi U_2)$ are standard normally distributed.

Step 3 Since we need random numbers that are normally distributed with zero mean and standard deviation of 0.008 we take the standard normally distributed random numbers, $Z_i, i = 1..N$ generated in step 2 and the transformation $X_i = 0.008Z_i$, for all i . The X_i are $N(0, 0.008)$.

The basis for step 2 is the following theorem:

Theorem 5.1.1. Let X be a random variable with probability density f on a set $A = \{x \in (R) : f(x) > 0\}$. The transformation $h : A \rightarrow B = h(A)$ be invertible and the inverse h^{-1} be continuous. Then $Y = h(X)$ has the density function

$$y \mapsto f(h^{-1}(y)) \left| \det\left(\frac{dh^{-1}}{dy}(y)\right) \right|, \quad y \in B.$$

Proof: of Step 3.

Let $Y \sim N(0, 1)$. Now, consider $X = \sigma Y + \mu$.

Expected Value: $E(X) \stackrel{(2.8)}{=} \sigma E(Y) + \mu = \mu$

Variance: $Var(X) \stackrel{(2.8)}{=} \sigma^2 Var(Y) = \sigma^2$

Thus $X \sim N(\mu, \sigma^2)$, which means for our model that we have generated a stochastic factor $\epsilon \sim N(0, 0.008^2)$. \square

5.2 Solving the model

As soon as we have generated our standard normally distributed pseudo random variables ϵ_t we can model a path for the development of labour over time.

Remember the dynamics for z are given by: $\ln z_{t+1} = \rho \ln z_t + \epsilon_t$

For computational reasons, we will work with the logarithm at the latest possible point during our analysis. Therefore we rewrite the equation by substituting $\ln z = y$. We get:

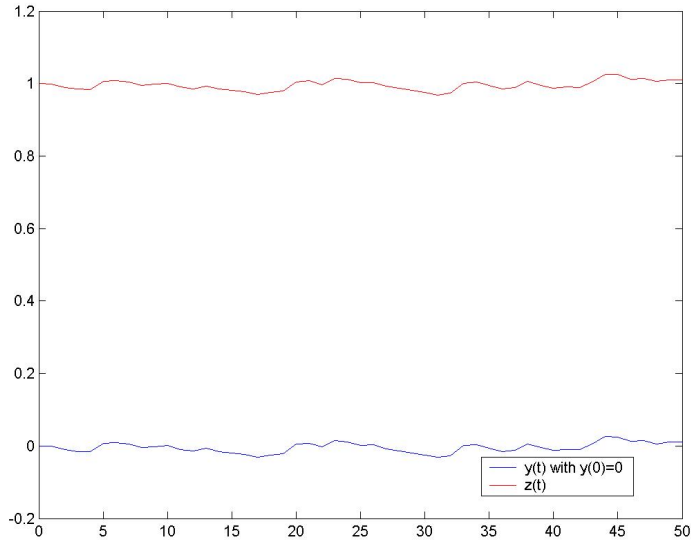
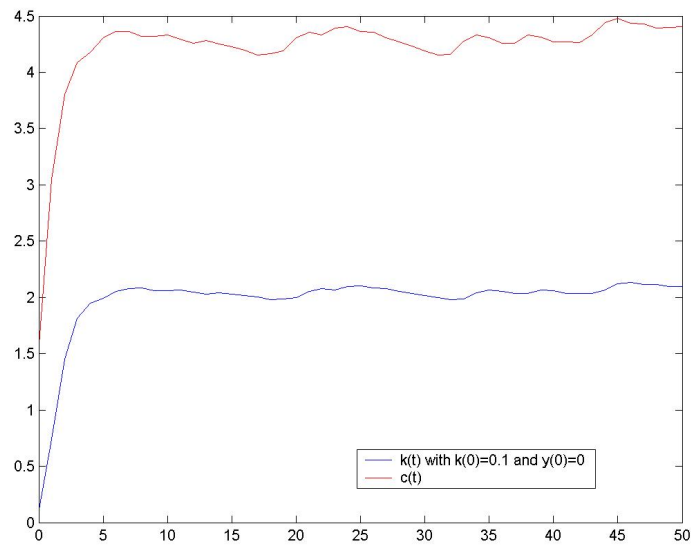
$$y_{t+1} = \rho y_t + \epsilon_t \tag{5.1}$$

$$k_{t+1} = e^{y_t} A k_t^\alpha - c_t \tag{5.2}$$

Starting with modelling the path for y we get the path for z by taking $z_t = e^{y_t}$ for $t = 1 \dots N$. Figure 5.1 shows a possible outcome for both y and z for an initial value $y(0) = 0$. See Appendix A.2 for the exact MATLAB code.

In the next step both path y and path c are modelled, since both are dependent of each other. As one can see in the formula for k , c is already included, but is one time step aback. Thus, if k is computed on the interval $[0, N]$, then c will just be computed from 0 up to the point $N - 1$. Since we know the optimal control function to our particular stochastic growth problem, we can calculate an initial value for c as long as we have chosen initial values for y and for z . Again the exact MATLAB code is given in Appendix A.2.. In figure 5.2 the initial values are $k(0) = 0.1$ and $y(0) = 0$.

From there it is easy to calculate the stochastic growth factor m . Remember

Figure 5.1: Paths for y and z Figure 5.2: Path for capital k and consumption c

(4.4):

$$m(c_t, c_{t+1}) = \beta \frac{c_t}{c_{t+1}}.$$

One path for $m(c_t, c_{t+1})$ is given in Figure 5.3 for the usual initial values used throughout this chapter. Looking again at chapter 3, where the asset price

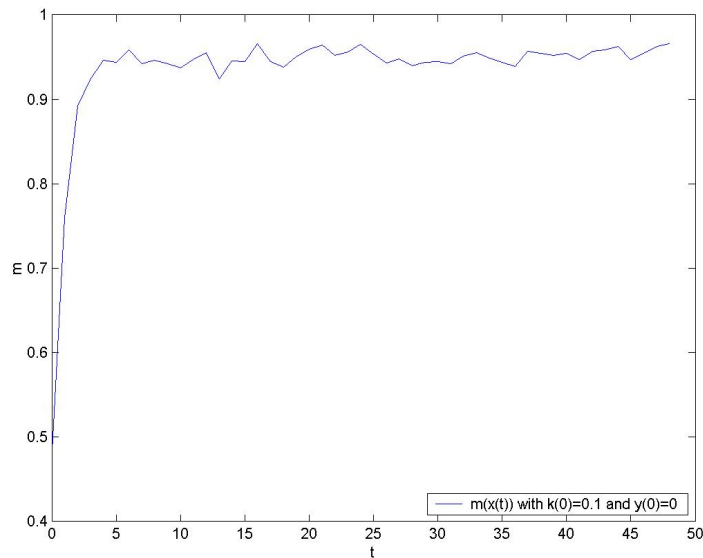


Figure 5.3: The stochastic discount factor m

characteristics in the context of a stochastic growth model were introduced, it can be seen, that we need the expected value and the standard deviation of m for computing the risk-free interest rate R_f , the Upper Bound S_B and the Sharpe Ratio SR , itself. See (3.3),(3.6),(3.7) and (3.9).

If we choose initial values $y(0) = 0$, $k(0) = 0.1$ and a time horizon of $N = 50$

we get:

$$E[m] = \frac{1}{N-1} \sum_{t=0}^{N-2} m(c_t, c_{t+1}) = \mathbf{0.9343} \quad (5.3)$$

$$\sigma_m = \sqrt{\frac{1}{N-1} \sum_{t=0}^{N-2} \text{Var}[m(c_t, c_{t+1})]} = \mathbf{0.0726} \quad (5.4)$$

$$R_f = \frac{1}{E[m]} = \mathbf{1.0704} \quad (5.5)$$

$$S_B = \frac{\sigma_m}{E[m]} = \mathbf{0.0760}. \quad (5.6)$$

In the formula for the expected value for m , it is only divided by $N-1$, because we just have N values for $c(x_t)$. Since we use c_{t+1} for calculating $m(c_t, c_{t+1})$ we always look one consumption step ahead of time and therefore yield $N-1$ values for m .

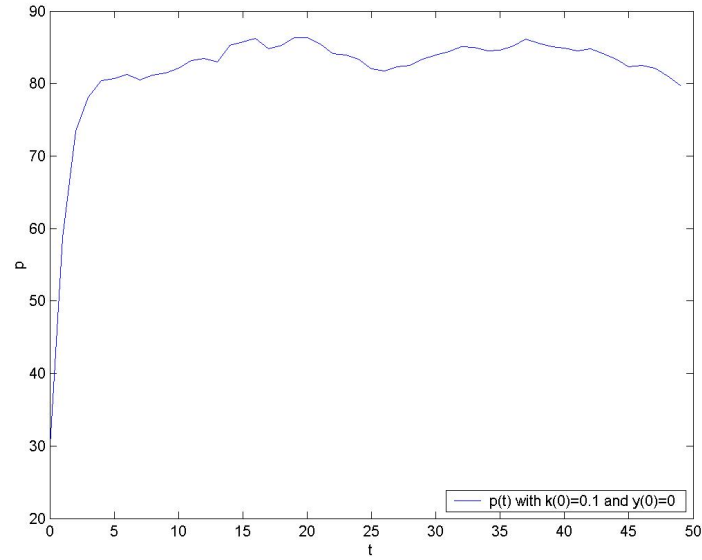
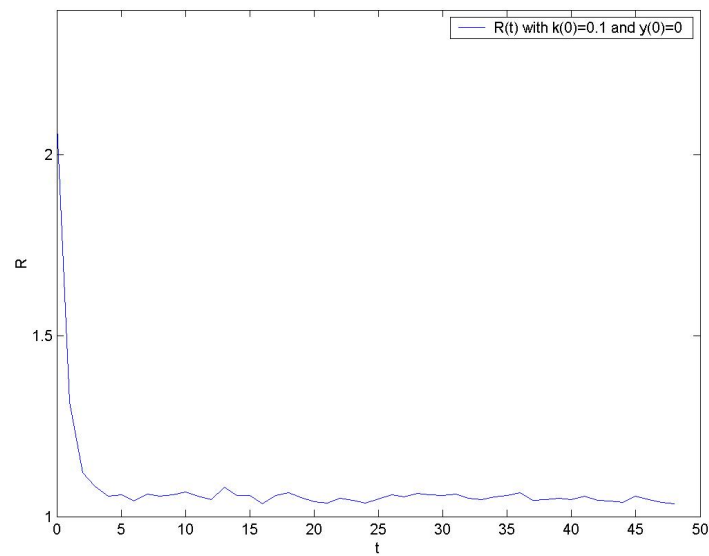
Consider the Sharpe Ratio as introduced in chapter 3. In order to compute the Sharpe Ratio we have to find the return path first. Therefore we have a look at the price formula of this problem, which is given by (4.5).

$$p_t = \frac{\beta}{1-\beta} (1-\alpha\beta) A z_t k_t^\alpha$$

Figure 5.4 shows a price path to our model with initial values $y(0) = 0$ and $k(0) = 0.1$. As soon as we have our prices to different points in time we can calculate the return. Since the wealth portfolio is a claim to future consumption the dividend d_t is given by consumption c_t . Therefore the gross return function becomes:

$$R_{t+1} = \frac{c_{t+1} + p_{t+1}}{p_t} \quad (5.7)$$

Figure 5.5 shows the path for the gross return for our consumption based model with initial values as used above. Now, we can start to evaluate the Sharpe Ratio for our model. Starting from R , we are able to compute the expectation of R over time, the covariance of gross return R and stochastic

Figure 5.4: The price p Figure 5.5: The gross return R

growth model m and finally the Sharpe Ratio.

$$E[R] = \frac{1}{N-1} \sum_{t=0}^{N-2} R_t = \mathbf{1.0815} \quad (5.8)$$

$$\sigma_R = \sqrt{\frac{1}{N-1} \sum_{t=0}^{N-2} Var[R_t]} = \mathbf{0.1473} \quad (5.9)$$

$$Cov[R, m] = \frac{1}{N-1} \sum_{t=0}^{N-2} Cov[R_t, m_t] = \mathbf{-0.0103} \quad (5.10)$$

$$SR = \frac{-R^f Cov[m, R]}{\sigma[R]} = \mathbf{0.0751}. \quad (5.11)$$

5.3 Sharpe Ratios subject to initial values

In this chapter we will give solutions for the Sharpe Ratio and relevant asset price characteristics subject to different initial values in Table 5.1. Remember the domain in which we compute the model is $\Omega = [0.1, 10] \times [-0.32, 0.32]$ and that the paths for y and k oscillate approximately around some "equilibrium" point $(0, 2)$ at a pretty early point in time. We have seen in Figure 4.1 that it already starts to oscillate when a time horizon of just $N = 50$ is chosen. Nonetheless even for any pair of initial values with $k = 2$ a time horizon of $N = 10000$ is needed to get to a final value for the Sharpe Ratio. For any other pair of initial values the maximum time horizon is chosen to be $N = 50000$, see Table 5.1. The MATLAB codes generating these results are given in appendix (A.2)

We find that the Sharpe Ratios differ from the Sharpe Ration given by Grüne and Semmler, [7], p.16, that is 0.007999.

Before we analyse the model for reasons for this difference in chapter 7, we will have a look at the random variable approach in chapter 6.

| N | Initial values: $(y_0, k_0) = (-0.32, 2.0)$ | | | | |
|----------------------|---|----------|----------|----------|----------|
| | 50 | 500 | 1000 | 5000 | 10000 |
| $E[m]$ | 0.9436 | 0.9494 | 0.9498 | 0.9500 | 0.9500 |
| σ_m | 0.0171 | 0.0103 | 0.0095 | 0.0088 | 0.0087 |
| R^f | 1.0597 | 1.0533 | 1.0529 | 1.0527 | 1.0526 |
| $E[R]$ | 1.0601 | 1.0534 | 1.0530 | 1.0527 | 1.0527 |
| σ_R | 0.0188 | 0.0115 | 0.0106 | 0.0098 | 0.0097 |
| $Cov[R, m]$ | -0.00032 | -0.00012 | -0.00010 | -0.00009 | -0.00008 |
| S_B | 0.0181 | 0.0109 | 0.0100 | 0.0092 | 0.0091 |
| Sharpe Ratio | 0.0181 | 0.0109 | 0.0100 | 0.0092 | 0.0091 |

| N | Initial values: $(y_0, k_0) = (0.32, 2.0)$ | | | | |
|----------------------|--|----------|----------|----------|----------|
| | 50 | 500 | 1000 | 5000 | 10000 |
| $E[m]$ | 0.9557 | 0.9506 | 0.9504 | 0.9501 | 0.9501 |
| σ_m | 0.0178 | 0.0104 | 0.0095 | 0.0088 | 0.0087 |
| R^f | 1.0464 | 1.0519 | 1.0522 | 1.0525 | 1.0526 |
| $E[R]$ | 1.0467 | 1.0521 | 1.0523 | 1.0526 | 1.0526 |
| σ_R | 0.0203 | 0.0117 | 0.0107 | 0.0098 | 0.0097 |
| $Cov[R, m]$ | -0.00036 | -0.00012 | -0.00010 | -0.00009 | -0.00008 |
| S_B | 0.0187 | 0.0110 | 0.0100 | 0.0092 | 0.0091 |
| Sharpe Ratio | 0.0186 | 0.0110 | 0.0100 | 0.0092 | 0.0091 |

| N | Initial values: $(y_0, k_0) = (0, 2.0)$ | | | | |
|----------------------|---|----------|----------|----------|----------|
| | 50 | 500 | 1000 | 5000 | 10000 |
| $E[m]$ | 0.9495 | 0.9500 | 0.9501 | 0.9500 | 0.9500 |
| σ_m | 0.0100 | 0.0092 | 0.0088 | 0.0086 | 0.0086 |
| R^f | 1.0531 | 1.0526 | 1.0526 | 1.0526 | 1.0526 |
| $E[R]$ | 1.0533 | 1.0527 | 1.0527 | 1.0527 | 1.0527 |
| σ_R | 0.0112 | 0.0102 | 0.0099 | 0.0096 | 0.0096 |
| $Cov[R, m]$ | -0.00011 | -0.00009 | -0.00008 | -0.00008 | -0.00008 |
| S_B | 0.0105 | 0.0096 | 0.0093 | 0.0091 | 0.0091 |
| Sharpe Ratio | 0.0105 | 0.0096 | 0.0093 | 0.0091 | 0.0091 |

| N | Initial values: $(y_0, k_0) = (-0.32, 9.0)$ | | | | | |
|---------------------|---|----------|----------|----------|----------|----------|
| | 50 | 500 | 1000 | 5000 | 10000 | 50000 |
| $E[m]$ | 0.9552 | 0.9506 | 0.9503 | 0.9501 | 0.9501 | 0.9500 |
| σ_m | 0.0711 | 0.0241 | 0.0181 | 0.0111 | 0.0099 | 0.0088 |
| R^f | 1.0468 | 1.0520 | 1.0523 | 1.0525 | 1.0526 | 1.0526 |
| $E[R]$ | 1.0509 | 1.0525 | 1.0525 | 1.0527 | 1.0527 | 1.0527 |
| σ_R | 0.0551 | 0.0199 | 0.0156 | 0.0110 | 0.0103 | 0.0097 |
| $Cov[R, m]$ | -0.0039 | -0.00047 | -0.00028 | -0.00012 | -0.00010 | -0.00008 |
| S_B | 0.0744 | 0.0254 | 0.0190 | 0.0117 | 0.0105 | 0.0093 |
| Sharpe Ratio | 0.0739 | 0.0250 | 0.0187 | 0.0115 | 0.0103 | 0.0092 |

| N | Initial values: $(y_0, k_0) = (0.32, 9.0)$ | | | | | |
|---------------------|--|----------|----------|----------|----------|----------|
| | 50 | 500 | 1000 | 5000 | 10000 | 50000 |
| $E[m]$ | 0.9662 | 0.9517 | 0.9509 | 0.9502 | 0.9501 | 0.9501 |
| σ_m | 0.0437 | 0.0170 | 0.0135 | 0.0098 | 0.0092 | 0.0086 |
| R^f | 1.0349 | 1.0508 | 1.0516 | 1.0524 | 1.0525 | 1.0526 |
| $E[R]$ | 1.0367 | 1.0511 | 1.0518 | 1.0525 | 1.0526 | 1.0527 |
| σ_R | 0.0395 | 0.0165 | 0.0135 | 0.0105 | 0.0100 | 0.0096 |
| $Cov[R, m]$ | -0.0017 | -0.00028 | -0.00018 | -0.00011 | -0.00009 | -0.00008 |
| S_B | 0.0452 | 0.0178 | 0.0142 | 0.0103 | 0.0097 | 0.0091 |
| Sharpe Ratio | 0.0450 | 0.0177 | 0.0140 | 0.0102 | 0.0096 | 0.0091 |

| N | Initial values: $(y_0, k_0) = (0, 9.0)$ | | | | | |
|---------------------|---|----------|----------|----------|----------|----------|
| | 50 | 500 | 1000 | 5000 | 10000 | 50000 |
| $E[m]$ | 0.9606 | 0.9511 | 0.9506 | 0.9501 | 0.9501 | 0.9501 |
| σ_m | 0.0558 | 0.0199 | 0.0153 | 0.0103 | 0.0095 | 0.0087 |
| R^f | 1.0410 | 1.0514 | 1.0520 | 1.0525 | 1.0525 | 1.0526 |
| $E[R]$ | 1.0437 | 1.0518 | 1.0522 | 1.0526 | 1.0526 | 1.0527 |
| σ_R | 0.0463 | 0.0177 | 0.0142 | 0.0107 | 0.0101 | 0.0096 |
| $Cov[R, m]$ | -0.0026 | -0.00035 | -0.00021 | -0.00011 | -0.00010 | -0.00008 |
| S_B | 0.0581 | 0.0209 | 0.0161 | 0.0108 | 0.0100 | 0.0092 |
| Sharpe Ratio | 0.0578 | 0.0207 | 0.0159 | 0.0107 | 0.0099 | 0.0091 |

| N | Initial values: $(y_0, k_0) = (-0.32, 0.1)$ | | | | | |
|---------------------|---|----------|----------|----------|----------|----------|
| | 50 | 500 | 1000 | 5000 | 10000 | 50000 |
| $E[m]$ | 0.9276 | 0.9478 | 0.9490 | 0.9498 | 0.9499 | 0.9500 |
| σ_m | 0.0660 | 0.0236 | 0.0178 | 0.0110 | 0.0099 | 0.0088 |
| R^f | 1.0780 | 1.0550 | 1.0538 | 1.0528 | 1.0527 | 1.0526 |
| $E[R]$ | 1.0869 | 1.0561 | 1.0543 | 1.0530 | 1.0528 | 1.0527 |
| σ_R | 0.1262 | 0.0423 | 0.0307 | 0.0162 | 0.0133 | 0.0104 |
| $Cov[R, m]$ | -0.0082 | -0.00097 | -0.00053 | -0.00017 | -0.00013 | -0.00009 |
| S_B | 0.0712 | 0.0249 | 0.0187 | 0.0116 | 0.0104 | 0.0093 |
| Sharpe Ratio | 0.0703 | 0.0243 | 0.0182 | 0.0112 | 0.0100 | 0.0091 |

| N | Initial values: $(y_0, k_0) = (0.32, 0.1)$ | | | | | |
|---------------------|--|---------|----------|----------|----------|----------|
| | 50 | 500 | 1000 | 5000 | 10000 | 50000 |
| $E[m]$ | 0.9140 | 0.9492 | 0.9496 | 0.9500 | 0.9500 | 0.9500 |
| σ_m | 0.0769 | 0.0259 | 0.0193 | 0.0115 | 0.0102 | 0.0089 |
| R^f | 1.0627 | 1.0536 | 1.0530 | 1.0527 | 1.0526 | 1.0526 |
| $E[R]$ | 1.0765 | 1.0550 | 1.0538 | 1.0529 | 1.0528 | 1.0527 |
| σ_R | 0.1710 | 0.0554 | 0.0398 | 0.0197 | 0.0155 | 0.0110 |
| $Cov[R, m]$ | -0.00130 | -0.0014 | -0.00074 | -0.00021 | -0.00015 | -0.00009 |
| S_B | 0.0817 | 0.0273 | 0.0203 | 0.0122 | 0.0107 | 0.0093 |
| Sharpe Ratio | 0.0805 | 0.0266 | 0.0196 | 0.0114 | 0.0100 | 0.0090 |

| N | Initial values: $(y_0, k_0) = (0, 0.1)$ | | | | | |
|---------------------|---|---------|----------|----------|----------|----------|
| | 50 | 500 | 1000 | 5000 | 10000 | 50000 |
| $E[m]$ | 0.9342 | 0.09485 | 0.9493 | 0.9499 | 0.9500 | 0.9500 |
| σ_m | 0.0710 | 0.0425 | 0.0184 | 0.0112 | 0.0100 | 0.0088 |
| R^f | 1.0704 | 1.0543 | 1.0534 | 1.0528 | 1.0527 | 1.0526 |
| $E[R]$ | 1.0815 | 1.0555 | 1.0541 | 1.0530 | 1.0528 | 1.0527 |
| σ_R | 0.1473 | 0.0483 | 0.0349 | 0.0178 | 0.0143 | 0.0106 |
| $Cov[R, m]$ | -0.0103 | -0.0012 | -0.00062 | -0.00019 | -0.00014 | -0.00009 |
| S_B | 0.0760 | 0.0258 | 0.0193 | 0.0118 | 0.0105 | 0.0093 |
| Sharpe Ratio | 0.0751 | 0.0252 | 0.0187 | 0.0112 | 0.0100 | 0.0091 |

Table 5.1: Sharpe Ratios depending on initial values

Chapter 6

Random Variable Approach

This chapter covers the random variable approach to our stochastic growth model. First, note, that all MATLAB codes used throughout this chapter can be found in appendix A.3.

6.1 Restrictions on ϵ

As one could see in chapter 4 the stochastic discount factor is given by:

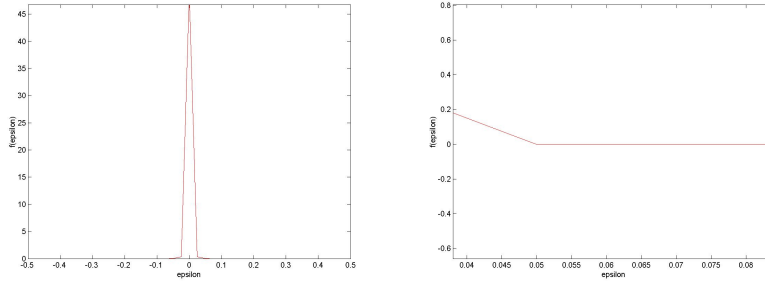
$$m(c_t, c_{t+1}) = \beta \frac{c_t}{c_{t+1}}$$

We will now look at m as a function of our random variable ϵ , which goes into the function via z and therefore c . Substituting the control function for c into the formula for m and just looking one step ahead from the initial values gives us:

$$m(k, y) = \frac{\beta e^{y_0} k_0^\alpha}{e^{\rho y_0 + \epsilon} \alpha \beta A e^{y_0} k_0^\alpha} \quad (6.1)$$

Again we have to find the Expected Value of m and several other characteristics in order to calculate the Upper Bound and the Sharpe Ratio.

If we want to calculate the expected value of our random variable m , we have

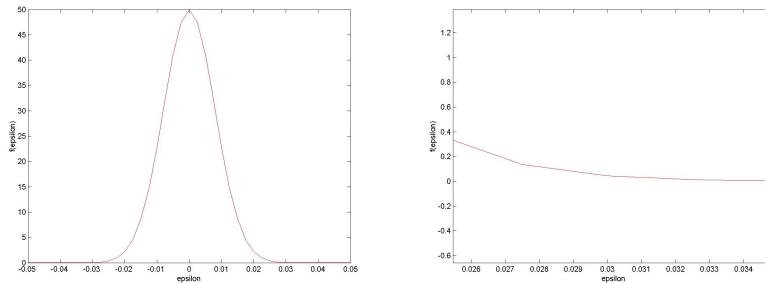
Figure 6.1: density function of ϵ in $[-0.5, 0.5]$

to use the density function of ϵ . Since $\epsilon \sim N(0, 0.008)$ the density function is:

$$\tilde{f}(\epsilon) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2} \frac{\epsilon^2}{0.008^2}} \quad (6.2)$$

Because of the fact that we work with MATLAB and that uses numerical methods for the calculation of integrals, we have to restrict the interval for ϵ . Figure 6.1 shows the density function. Obviously the numbers in the tail of the figure are very close to zero. A cutout of the right tail is given in Figure 6.2. This figure gives us the impression that all values outside the interval $[-0.05, 0.05]$ are approximately 0. In order to check that impression we also calculate the density function in the interval $[-0.05, 0.05]$ and look at those tails.

Thus we see that using the interval $[-0.032, 0.032]$ is sufficient for our calculation. We will use this interval, since it was also used in Grüne and

Figure 6.2: density function of ϵ in $[-0.05, 0.05]$

Semmler, [7], p. 15. If we broaden the interval the values we get for the Sharpe Ratio and other characteristics do not change any more.

6.2 Sharpe Ratios subject to initial values

Since we compute the following integrals only in the interval $[-0.032, 0.032]$, we have to scale the density function of ϵ by $\int_{-0.032}^{0.032} \tilde{f}(\epsilon) d\epsilon = 0.9999$ for $\sigma_\epsilon = 0.008$. We get:

$$f(\epsilon) = \frac{1}{0.9999} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2} \frac{\epsilon^2}{0.008^2}} \quad (6.3)$$

Thus the expected value for m is calculated holding c_0, k_0 and z_0 constant and by substituting the corresponding formulas for k_1 and z_1 where needed:

$$E[m] = \int_{-0.032}^{0.032} \frac{\beta e^{y_0} k_0^\alpha}{e^{\rho y_0 + \epsilon} \alpha \beta A e^{y_0} k_0^\alpha} f(\epsilon) d\epsilon \quad (6.4)$$

Calculation of the Variance of m :

$$Var[m] = \int_{-0.032}^{0.032} \left(\frac{\beta e^{y_0} k_0^\alpha}{e^{\rho y_0 + \epsilon} \alpha \beta A e^{y_0} k_0^\alpha} - E[m] \right)^2 f(\epsilon) d\epsilon \quad (6.5)$$

Calculation of the asset's expected return and variance:

$$E[R] = \int_{-0.032}^{0.032} \frac{c_1 + p_1}{p_0} f(\epsilon) d\epsilon \quad (6.6)$$

$$= \int_{-0.032}^{0.032} \frac{e^{\rho y_0 + \epsilon} (\alpha \beta e^{y_0} A k_0^\alpha)^\alpha}{\beta e^{y_0} k_0^\alpha} f(\epsilon) d\epsilon \quad (6.7)$$

$$Var[R] = \int_{-0.032}^{0.032} \left(\frac{e^{\rho y_0 + \epsilon} (\alpha \beta e^{y_0} A k_0^\alpha)^\alpha}{\beta e^{y_0} k_0^\alpha} - E[R] \right)^2 f(\epsilon) d\epsilon \quad (6.8)$$

The risk-free interest rate R_f , the Sharpe Ratio SR and the upper bound SB are calculated as usual. Therefore we need to compute the covariance of R

and m , as well.

$$Cov[R, m] = \int_{-0.032}^{0.032} \left(\frac{c_1 + p_1}{p_0} - E[R] \right) \left(\beta \frac{c_0}{c_1} - E[m] \right) f(\epsilon) d\epsilon \quad (6.9)$$

$$= \int_{-0.032}^{0.032} \left(\frac{e^{\rho y_0 + \epsilon} (\alpha \beta e^{y_0} A k_0^\alpha)^\alpha}{\beta e^{y_0} k_0^\alpha} - E[R] \right) \quad (6.10)$$

$$\cdot \left(\frac{\beta e^{y_0} k_0^\alpha}{e^{\rho y_0 + \epsilon} \alpha \beta A e^{y_0} k_0^\alpha} - E[m] \right) f(\epsilon) d\epsilon \quad (6.11)$$

Table 6.1 shows the characteristics depending on various initial values. The MATLAB codes generating these results are given in appendix (A.2).

In the Table all numbers are only calculated up to the fourth decimal digit. The more exact Sharpe Ratio is 0.00799533412689, which fits nicely with the result as can be found in Grüne and Semmler, [7], p.16, that is 0.007999.

| Initial values: $k(0) = 0.1$ | | | |
|------------------------------|------------|----------------|---------------|
| | $y(0) = 0$ | $y(0) = -0.32$ | $y(0) = 0.32$ |
| $E[m]$ | 0.4815 | 0.5199 | 0.4459 |
| σ_m | 0.0038 | 0.0042 | 0.0036 |
| R^f | 2.0770 | 1.9235 | 2.2428 |
| $E[R]$ | 2.0772 | 1.9236 | 2.2430 |
| σ_R | 0.0166 | 0.0154 | 0.0179 |
| $Cov[R, m]$ | -0.00006 | -0.00006 | -0.00006 |
| S_B | 0.0080 | 0.0080 | 0.0080 |
| SR | 0.0080 | 0.0080 | 0.0080 |

| Initial values: $k(0) = 9.0$ | | | |
|------------------------------|------------|----------------|---------------|
| | $y(0) = 0$ | $y(0) = -0.32$ | $y(0) = 0.32$ |
| $E[m]$ | 1.3216 | 1.4271 | 1.2239 |
| σ_m | 0.0106 | 0.0114 | 0.0098 |
| R^f | 0.7567 | 0.7007 | 0.8171 |
| $E[R]$ | 0.7567 | 0.7008 | 0.8171 |
| σ_R | 0.0061 | 0.0056 | 0.0065 |
| $Cov[R, m]$ | -0.00006 | -0.00006 | -0.00006 |
| S_B | 0.0080 | 0.0080 | 0.0080 |
| SR | 0.0080 | 0.0080 | 0.0080 |

| Initial values: $\mathbf{k}(0) = 2.0$ | | | |
|---------------------------------------|------------------------------|----------------|---------------|
| | $\mathbf{y}(0) = \mathbf{0}$ | $y(0) = -0.32$ | $y(0) = 0.32$ |
| $E[m]$ | 0.9430 | 1.0183 | 0.8733 |
| σ_m | 0.0075 | 0.0081 | 0.0070 |
| R^f | 1.0604 | 0.9821 | 1.1451 |
| $E[R]$ | 1.0605 | 0.9821 | 1.1452 |
| σ_R | 0.0085 | 0.0079 | 0.0092 |
| $Cov[R, m]$ | -0.00006 | -0.00006 | -0.00006 |
| S_B | 0.0080 | 0.0080 | 0.0080 |
| SR | 0.0080 | 0.0080 | 0.0080 |

Table 6.1: Sharpe Ratios depending on initial values

Chapter 7

Comparison

As one can see by comparing the results of chapter 5 and chapter 6, Table 5.1 and Table 6.1, the results differ rather a lot. At first we have to note that the only results we can really compare and which we hoped to be equal are the Sharpe Ratio and the upper bound. We do now have to admit that both approaches have rather different outcomes.

Consider the initial values $(y_0, k_0) = (0, 0.1)$. Then the result for example for $E[m]$ is 0.4815 when using the random variable approach and the result is 0.9500 for the time series with $N = 10000$. It is important to mention that we cannot compare those results with each other since in the time series approach we middle over a lot of results in the long run, that oscillate at a pretty early point in time approximately around $(y, k) \sim (0, 2)$. Thus we can compare the outcome of $E[m]$ from the time series only with the outcome of the random variable for initial values $(y_0, k_0) = (0, 2)$. Same holds for R_f and $E[R]$.

Nonetheless the results obviously do not fit, therefore we need to take a more detailed look at the programmes and the differences between both approaches. We want to begin this analysis by checking the time series programme for correctness. After that we will have a look at the correlations between the random variables in both programmes.

7.1 Programme Check

We will now verify all programmes needed for the time series approach. The codes are given in Appendix (A.2).

At first we want to check if the equations for the time series for y , k and c work properly. One way to do so is to rewrite some of the equations. Assume that we just generate $N = 10000$ results for y_1, k_1 and c_1 , i.e. we look at sample data just one time step ahead from the initial values. According to the central limit theorem (2.1.18), averaging over these values should give us approximately the same values for y , k and c as we would get by using the random variable approach, which works by finding the expected value in one time step. The Matlab code for simulating the sample data are given in the programme SR_control, (A.4)

In order to generate the sample data we leave the programmes RWfibonacci.m and boxmuller.m the way they are, because we still need those $N(0, 0.008^2)$ distributed pseudo random variables ϵ . But we will change the routine in Refz, since we do not need a path over time but 10.000 results for the first time step. Thus we drop the time subscript t and use the subscript j to denote the "j-th sample data". We rewrite the equation for y_{j+1} in the following way:

$$y_{j+1} = \rho y_0 + \epsilon_j \quad (7.1)$$

Note, that the parameter t is not a time subscript any more, but rather an index denoting the number of an element of the sample data.

After having generated the sample data for y , we will now have a closer look at the formulas for k and c . Remember the original formulas:

$$\begin{aligned} k_{j+1} &= e^{y_j} A k_j^\alpha - c_j \\ c_j &= (1 - \alpha\beta) A e^{y_j} k_j^\alpha \end{aligned}$$

For our problem those formulas become:

$$k_{j+1} = e^{y_0} A k_0^\alpha - c_0 \quad (7.2)$$

$$c_{j+1} = (1 - \alpha\beta) A e^{y_j} k_1^\alpha \quad (7.3)$$

Note that the randomness just comes into the formula for c_{j+1} by y_j , because for the calculation of k_{j+1} all parameters and initial values are fix. Thus we will not have to calculate sample data for k . The sample data for c_{j+1} are

given by the above formula.

After obtaining all our datas we can now look at our asset price characteristics. At first we must find an expected value for m . The equation for m changes slightly in order to get sample data for m .

$$m_j = \beta \frac{c_0}{c_{j+1}} \quad (7.4)$$

$$E[m] = \frac{1}{N} \sum_{j=1}^N m_j \quad (7.5)$$

The next adjustment we have to make is in the return formula. The price formula can be used in the same way we used it beforehand. Based on the same principle the return equation becomes:

$$R_j = \frac{c_{j+1} + p_{j+1}}{p_0} \quad (7.6)$$

$$E[R] = \frac{1}{N} \sum_{j=1}^N R_j \quad (7.7)$$

These are all changes we have to make. From here we can use the original formulas for the calculation of the Sharpe Ratio. Table 7.1 shows the results for Sharpe Ratios and other financial characteristics, such as the risk-free interest rate, again dependent on different initial values. By comparison of this tabular with Table 6.1 that gives the results for the random variable approach, one can see, that all values are nearly identical, which tells us, that the programme set-up in general was correct. It also tells us that the pseudo random variables cannot be held responsible for the differences in the outcomes we detected beforehand.

Thus we have to verify both approaches for differences.

| Initial values: $k(0) = 0.1$ | | | |
|------------------------------|------------|----------------|---------------|
| | $y(0) = 0$ | $y(0) = -0.32$ | $y(0) = 0.32$ |
| $E[m]$ | 0.4814 | 0.5198 | 0.4458 |
| R^f | 2.0772 | 1.9237 | 2.2430 |
| $E[R]$ | 2.0774 | 1.238 | 2.2432 |
| \mathbf{S}_B | 0.0080 | 0.0080 | 0.0080 |
| \mathbf{SR} | 0.0080 | 0.0080 | 0.0080 |

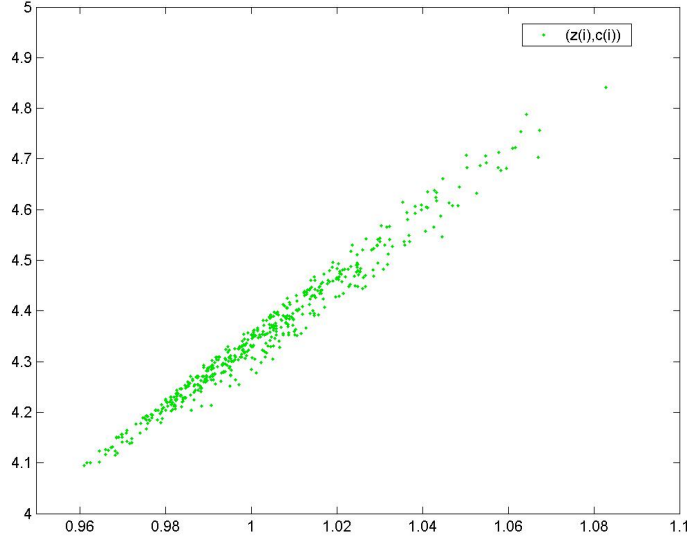
| Initial values: $k(0) = 9.0$ | | | |
|------------------------------|------------|----------------|---------------|
| | $y(0) = 0$ | $y(0) = -0.32$ | $y(0) = 0.32$ |
| $E[m]$ | 1.3214 | 1.4269 | 1.2238 |
| R^f | 0.7567 | 0.7008 | 0.8172 |
| $E[R]$ | 0.7568 | 0.7008 | 0.8172 |
| \mathbf{S}_B | 0.0080 | 0.0080 | 0.0080 |
| \mathbf{SR} | 0.0080 | 0.0080 | 0.0080 |

| Initial values: $\mathbf{k}(0) = \mathbf{2.0}$ | | | |
|--|------------------------------|----------------|---------------|
| | $\mathbf{y}(0) = \mathbf{0}$ | $y(0) = -0.32$ | $y(0) = 0.32$ |
| $E[m]$ | 0.9429 | 1.0182 | 0.8732 |
| R^f | 1.0606 | 0.9821 | 1.1452 |
| $E[R]$ | 1.0606 | 0.9822 | 1.1453 |
| \mathbf{S}_B | 0.0080 | 0.0080 | 0.0080 |
| \mathbf{SR} | 0.0080 | 0.0080 | 0.0080 |

Table 7.1: Sharpe Ratios as an outcome of the control programme SR_control

7.2 Correlations

The first thing we will do in this analysis is looking at the correlations between the random variables and compare the results of both programmes. In this context the correlation coefficient indicates the strength and direction of a *linear* relationship between two random variables. Thus correlation refers to the departure of two variables from independence. As one can easily see by looking at the formula for consumption there is an interdependence of y, k^α and c . The bivariate plot given in Figure 7.1 shows the linear dependence of z and c . If one tries to fit a single straight line through the dots it would have a positive slope. Thus we would also expect a positive relationship between those parameters. The correlation coefficient $\rho_{X,Y}$ between two

Figure 7.1: point pairs (z,c)

random variables was already defined in chapter 2.1 in definition 2.1.10. We can use this formula for the random variable approach. For the time series model the formula becomes:

$$\rho_{z,c} = \frac{\frac{1}{N} \sum_{t=0}^{N-1} ((z_t - E[z])(c_t - E[c]))}{\sqrt{\frac{1}{N} \sum_{t=0}^{N-1} ((z_t - E[z])^2)} \sqrt{\frac{1}{N} \sum_{t=0}^{N-1} ((c_t - E[c])^2)}} \quad (7.8)$$

This equation obviously holds for all other time series as well. Table 7.2 shows correlations in the time series model.

Let us now look at the correlations in the random variable model. Note again, that this approach only looks one time step ahead from the initial values when calculating the expectations, variances and covariances. It just integrates by using the density function of ϵ . Remember the formula for k_1 .

$$k_1 = e^{y_0} A k_0^\alpha - c_0$$

It is obvious that there is no stochastic influence. All values that are needed for its calculation are initial values. Thus the expected value $E[k_1]$ is equal

| Initial values: $k(0) = 0.1$ | | | |
|------------------------------|------------|----------------|---------------|
| $Cor[\cdot, \cdot]$ | $y(0) = 0$ | $y(0) = -0.32$ | $y(0) = 0.32$ |
| $[z, c]$ | 0.9608 | 0.9832 | 0.9226 |
| $[z, p]$ | 0.9608 | 0.9832 | 0.9226 |
| $[z, k]$ | 0.8309 | 0.8918 | 0.7476 |
| $[k^\alpha, c]$ | 0.8894 | 0.9126 | 0.8682 |

| Initial values: $k(0) = 9.0$ | | | |
|------------------------------|------------|----------------|---------------|
| $Cor[\cdot, \cdot]$ | $y(0) = 0$ | $y(0) = -0.32$ | $y(0) = 0.32$ |
| $[z, c]$ | 0.9634 | 0.9634 | 0.9664 |
| $[z, p]$ | 0.9634 | 0.9634 | 0.9664 |
| $[z, k]$ | 0.5845 | 0.5023 | 0.7456 |
| $[k^\alpha, c]$ | 0.8922 | 0.8379 | 0.9549 |

| Initial values: $\mathbf{k(0) = 2.0}$ | | | |
|---------------------------------------|------------|----------------|---------------|
| $Cor[\cdot, \cdot]$ | $y(0) = 0$ | $y(0) = -0.32$ | $y(0) = 0.32$ |
| $[z, c]$ | 0.9882 | 0.9886 | 0.9871 |
| $[z, p]$ | 0.9882 | 0.9886 | 0.9871 |
| $[z, k]$ | 0.8940 | 0.8973 | 0.8873 |
| $[k^\alpha, c]$ | 0.9518 | 0.9527 | 0.9487 |

Table 7.2: Correlations depending on initial values

to k_1 itself, which means that its variance is 0. This in turn does not allow us to calculate the correlations between k and other variables, because we would have to divide by zero. Whenever it is not possible to compute the correlation we will look at the covariance, since it already tells us if there is any kind of relationship between those variables.

Since the degree of the relationship between all parameters does not change with the initial values, Table 7.3 shows all correlations and covariances that turn up in this model.

As we have already expected by looking at the random variable model set-up, there is no linear relationship in the stochastic sense of k and z , and k^α and c . This is the main difference between both models, because for every pair of initial values there is always a positive linear relationship between those variables in the time series approach. At a closer look one even finds that in every case, there is not only a relationship but also a strong one. Hence the results for the Sharpe Ratio differ between both programmes.

| | |
|---------------------|--------------|
| $Cor[\cdot, \cdot]$ | (y_0, k_0) |
| $[z, c]$ | 1.0000 |
| $[z, p]$ | 1.0000 |
| $Cov[\cdot, \cdot]$ | |
| $[z, k]$ | 0 |
| $[k^\alpha, c]$ | 0 |

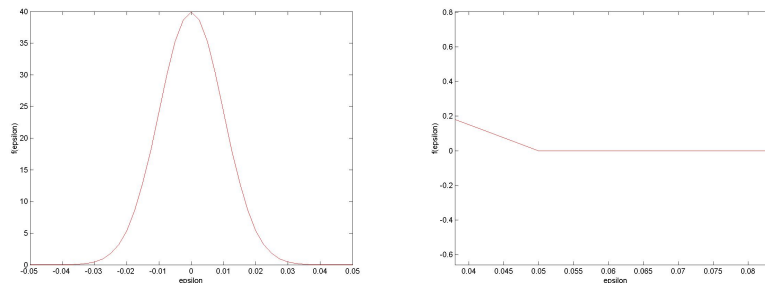
Table 7.3: Correlations and Covariances in the RV model

Considering that the influences vary that much between those models it is remarkable that the Sharpe Ratios are still quite close to each other. The small variance of ϵ might account for this fact.

7.3 σ_ϵ and Model Outcomes

In order to figure out what impact the chosen variance of ϵ does have on both models and especially on the different outcomes we will now vary the variance of ϵ in both programmes. Before we can calculate the Sharpe Ratios we have to figure out the interval for the random variable approach. Figure 7.2 shows the density function, when $\sigma_\epsilon = 0.01$. Thus we take the interval $[-0.05, 0.05]$ for all calculations in the random variable approach.

The following Table 7.4 shows the Sharpe Ratio and other characteristics, when $\epsilon \sim N(0, 0.01)$ and the time horizon N in the time series approach is set to 50000.

Figure 7.2: density function of ϵ in $[-0.05, 0.05]$, $\sigma_\epsilon = 0.01$

| | Initial Values | | | | | |
|--------------|----------------|----------|------------|----------|-----------|----------|
| | (0, 2) | | (-0.32, 2) | | (0.32, 2) | |
| | RV | TS | RV | TS | RV | TS |
| $E[m]$ | 0.9430 | 0.9501 | 1.0183 | 0.9501 | 0.8733 | 0.9501 |
| R_f | 1.0604 | 1.0526 | 0.9820 | 1.0526 | 1.1451 | 1.0526 |
| $E[R]$ | 1.0605 | 1.0527 | 0.9821 | 1.0527 | 1.1452 | 1.0527 |
| $Cov[R, m]$ | -0.00010 | -0.00013 | -0.00010 | -0.00013 | -0.00010 | -0.00013 |
| SB | 0.0100 | 0.0112 | 0.0100 | 0.0112 | 0.0100 | 0.0115 |
| SR_{cov} | 0.0100 | 0.0112 | 0.0100 | 0.0112 | 0.0100 | 0.0113 |
| SR_{stand} | 0.0099 | 0.0112 | 0.0099 | 0.0112 | 0.0099 | 0.0113 |

| | Initial Values | | | | | |
|--------------|----------------|----------|----------|----------|------------|----------|
| | (0, 0.1) | | (0, 9) | | (-0.32, 9) | |
| | RV | TS | RV | TS | RV | TS |
| $E[m]$ | 0.4815 | 0.9500 | 1.3216 | 0.9501 | 1.4271 | 0.9501 |
| R_f | 2.0770 | 1.0526 | 0.7567 | 1.0526 | 0.7007 | 1.0526 |
| $E[R]$ | 2.0772 | 1.0527 | 0.7567 | 1.0527 | 0.7008 | 1.0527 |
| $Cov[R, m]$ | -0.00010 | -0.00013 | -0.00010 | -0.00013 | -0.00010 | -0.00013 |
| SB | 0.0100 | 0.0115 | 0.0100 | 0.0114 | 0.0100 | 0.0114 |
| SR_{cov} | 0.0100 | 0.0113 | 0.0100 | 0.0113 | 0.0100 | 0.0114 |
| SR_{stand} | 0.0099 | 0.0113 | 0.0099 | 0.0113 | 0.0099 | 0.0114 |

Table 7.4: Results for the Sharpe Ratio, $\sigma_\epsilon = 0.01$, $N = 50000$, $\epsilon \in [-0.05, 0.05]$

We notice that the results for both approaches are different to the results we found in chapter 5 and 6 in Table 5.1 and 6.1. For both models the Sharpe Ratios are higher. Thus changing the variance to a higher level seems to result in higher Sharpe Ratios. Looking at values in Table 7.4 more thoroughly we also find that that all characteristics, except the upper bound and the Sharpe Ratio, are equal to the numbers we found in chapter 5 and 6. That means the reason for the different results can only be found in the standard deviations of R and m . Consider for example the initial values $(y_0, k_0) = (0, 2)$. With $\sigma_\epsilon = 0.008$ we find $\sigma_m = 0.0085$ and $\sigma_R = 0.0095$ and with $\sigma_\epsilon = 0.01$ we find $\sigma_m = 0.0106$ and $\sigma_R = 0.0119$. Looking at the formulas (3.6),(3.7) and (3.9). we can account the change in the standard deviations for the change in the results for the upper bound and the Sharpe Ratio.

In order to analyse the impact of the standard deviation σ_ϵ on the Sharpe Ratios thoroughly more results are needed. Table 7.5 shows the results for different standard deviations.

| | $\sigma_\epsilon = 0.008$ $\epsilon \in [-0.032, 0.032]$ | | | $\sigma_\epsilon = 0.01$ $\epsilon \in [-0.05, 0.05]$ | |
|--------|---|--------|--------|--|--------|
| | RV | TS | | RV | TS |
| $E[m]$ | 0.9430 | 0.9500 | $E[m]$ | 0.9430 | 0.9501 |
| R^f | 1.0604 | 1.0526 | R^f | 1.0604 | 1.0526 |
| $E[R]$ | 1.0605 | 1.0527 | $E[R]$ | 1.0605 | 1.0527 |
| S_B | 0.0080 | 0.0091 | S_B | 0.0100 | 0.0112 |
| SR | 0.0080 | 0.0091 | SR | 0.0099 | 0.0112 |

| | $\sigma_\epsilon = 0.05$ $\epsilon \in [-0.25, 0.25]$ | | | $\sigma_\epsilon = 0.10$ $\epsilon \in [-0.45, 0.45]$ | |
|--------|--|--------|--------|--|--------|
| | RV | TS | | RV | TS |
| $E[m]$ | 0.9441 | 0.9515 | $E[m]$ | 0.9477 | 0.9559 |
| R^f | 1.0592 | 1.0510 | R^f | 1.0552 | 1.0462 |
| $E[R]$ | 1.0618 | 1.0543 | $E[R]$ | 1.0658 | 1.0595 |
| S_B | 0.0500 | 0.0550 | S_B | 0.1002 | 0.1076 |
| SR | 0.0499 | 0.0548 | SR | 0.0992 | 0.1064 |

| | $\sigma_\epsilon = 0.5$ $\epsilon \in [-2.5, 2.5]$ | | | $\sigma_\epsilon = 0.05$ $\epsilon \in [-5, 5]$ | |
|--------|---|--------|--------|--|--------|
| | RV | TS | | RV | TS |
| $E[m]$ | 1.0685 | 1.0880 | $E[m]$ | 1.5546 | 1.5242 |
| R^f | 0.9359 | 0.9191 | R^f | 0.6432 | 0.6561 |
| $E[R]$ | 1.2017 | 1.2660 | $E[R]$ | 1.7484 | 2.3957 |
| S_B | 0.5329 | 0.4724 | S_B | 1.3095 | 0.8752 |
| SR | 0.4151 | 0.3619 | SR | 0.4827 | 0.3428 |

Table 7.5: Sharpe Ratios depending on σ_ϵ , $N = 50000$, $(y_0, k_0) = (0, 2)$

So far, we are not able to compare the results with each other. Remember that we want to find out what impact has the standard deviation of ϵ on the difference between the results for the Sharpe Ratios in both approaches. We will therefore look at the outcome ratios.

| σ_ϵ | Ratio $\frac{SR_{RV}}{SR_{TS}}$ |
|-------------------|---------------------------------|
| 0.008 | 0.8791 |
| 0.01 | 0.8839 |
| 0.05 | 0.5030 |
| 0.1 | 0.9323 |
| 0.5 | 1.1470 |
| 1 | 1.4081 |

Table 7.6: Sharpe Ratio ratios

For the first four ratios we get the impression that the higher we choose the standard deviation for ϵ the closer the outcomes get to each other. The

smaller Sharpe Ratio of the random variable approach seems to react stronger on the higher standard deviation than the Sharpe Ratio of the random variable model. But as soon as we choose σ_ϵ to be "really" high we observe that the gap between the outcome of the time series approach and of the random variable becomes larger again and finally percentage wise is even larger than it was at the beginning. Thus for high standard deviations the impact of σ_ϵ is stronger on the time series approach than on the random variable approach.

Chapter 8

Asset Pricing with Loss Aversion

Since reality shows that consumption based models do not fit financial market characteristics to time series data, in this chapter we will adjust our stochastic growth model for loss aversion. The main idea behind asset pricing models with loss aversion is, that someone who has experienced much losses on risky assets in his past, will become even more careful with handling risky assets in the future than he has already been beforehand. Thus we will now look at a model taking one's past experiences into consideration. The model we introduce in this chapter is based on the paper by Grüne and Semmler, [8]. As for the basic consumption based model they used the stochastic dynamic programming approach to solve for Sharpe Ratios.

8.1 Our Reference Model and Loss Aversion

Look back at the model set-up of our reference stochastic growth in chapter 4. We will now adjust that model for the loss aversion. Thus, in this chapter we want to maximize over

$$E_t\left[\sum_{t=0}^{\infty}(\beta^t u(c_t) + b_t \beta^{t+1} \nu(X_{t+1}, S_t, r_t))\right] \quad (8.1)$$

$u(c_t)$: denotes the utility over consumption. As usual we take $u(c) = \ln c$.

β : is again the discount factor

X_{t+1} : is the change of wealth

S_t : the value of the agent's risky assets

r_t : a variable, measuring the agent's gains or losses prior to period t , denoted as a fraction of S_t . The economic interpretation of z_t is that it expresses the way of how the agent has experienced gains or losses in the past affecting his or her willingness to take risks.

R_t : is again the return on the risky asset

R^f : is still the risk-free interest rate as the case may be the return on a risk-free asset.

More precisely we have:

1. The change in wealth is calculated as

$$X_{t+1} = S_t R_t - S_t R^f \quad (8.2)$$

The difference $R^f - R_t$ can be positive, zero or negative.

2. Although r_t can be greater, equal or smaller than one, with $r_t = 1$ and thus with

$$\nu(X_{t+1}, S_t, 1) = \begin{cases} X_{t+1} & \text{for } X_{t+1} \geq 0 \\ \lambda X_{t+1} & \text{for } X_{t+1} < 0. \end{cases} \quad (8.3)$$

$\lambda > 1$ defined as follows:

$$\lambda(z_t) = \lambda + k(r_t - 1), \quad (8.4)$$

$k > 0$. This models the fact that a loss is more severe than a gain. For r_{t+1} we have:

$$r_{t+1} = \eta z_t \frac{\bar{R}}{R_{t+1}} + (1 - \eta), \quad (8.5)$$

$\eta \in [0, 1]$ and \bar{R} fixed, denoting the long time average of the risk-free interest rate.

3. The last parameter we have to discuss is b_t , which is related to aggregate consumption \tilde{c}_t in the following way:

$$b_t = b_0 \tilde{c}_t^{-1}, \quad (8.6)$$

\tilde{c}_t will be specified so as to hold the price-dividend ratio and the risky asset premium stationary. One important parameter is b_0 , which points the relevance of financial wealth on utility gains or losses relative to consumption out. With $b_0 = 0$ we obtain the usual consumption based model.

This is our reference model adjusted for loss aversion. Now, we have to take a closer look at the formulas that we need for the calculation of the asset price characteristics.

8.2 Loss Aversion and Financial Measures

Since our maximization problem changed due to the adjustment for risk aversion, we would also expect our formulas to change at least slightly. Since the utility function remains unchanged, the stochastic discount factor m for the risk free interest rate does not change. We still have:

$$m_{f,t+1} = \beta \frac{\tilde{c}_t}{\tilde{c}_{t+1}}$$

Thus R^f remains $\frac{1}{E[m]}$. The first considerable change can be found in a characterization for the risky asset. Remember in the consumption based model we had $E[mR] = 1$ for the same reason that $E[mR^f] = 1$, see chapter 3 (3.3). Now, we have:

$$1 = \underbrace{\beta E_t \left[R_{t+1} \frac{\tilde{c}_t}{\tilde{c}_{t+1}} \right]}_{=E_t[m_{f,t+1}R_{t+1}]} + b_0 \beta E_t [\hat{\nu} (R_{t+1}, r_t)] \quad (8.7)$$

with

$$\hat{\nu}(R_{t+1}, r_t) = \begin{cases} R_{t+1} - R_t^f, & R_{t+1} \geq r_t R_t^f \text{ and } r_t \leq 1 \\ (r_t - 1)R_t^f + \lambda(R_{t+1} - r_t R_t^f), & R_{t+1} < r_t R_t^f \text{ and } r_t \leq 1 \\ R_{t+1} - R_t^f, & R_{t+1} \geq R_t^f \text{ and } r_t > 1 \\ \lambda(r_t)(R_{t+1} - R_t^f), & R_{t+1} < R_t^f \text{ and } r_t > 1 \end{cases}$$

The second term of equation (1.7) expresses the risk of greater losses the agent is exposed to if he or she consumes less today and invests in risky assets instead of it.

Using that in this model the dividend d_{t+1} is chosen equal to \tilde{c}_{t+1} as before-hand, and then using the return function for the risky asset, that is:

$$R_{t+1} = \frac{p_{t+1} + c_{t+1}}{p_t}.$$

Thus our basic pricing equations becomes:

$$P_t = \begin{cases} E_t \left[\frac{m_{f,t+1} + \beta b_0}{1 + \beta b_0 R_t^f} (c_{t+1} + P_{t+1}) \right] & \text{for } R_{t+1} \geq r_t R_t^f \text{ and } r_t \leq 1 \\ E_t \left[\frac{m_{f,t+1} + \beta b_0 \lambda}{1 + \beta b_0 ((\lambda - 1)r_t + 1) R_t^f} (c_{t+1} + P_{t+1}) \right] & \text{for } R_{t+1} < r_t R_t^f \text{ and } r_t \leq 1 \\ E_t \left[\frac{m_{f,t+1} + \beta b_0}{1 + \beta b_0 R_t^f} (c_{t+1} + P_{t+1}) \right] & \text{for } R_{t+1} \geq R_t^f \text{ and } r_t > 1 \\ E_t \left[\frac{m_{f,t+1} + \beta b_0 \lambda(r_t)}{1 + \beta b_0 \lambda(r_t) R_t^f} (c_{t+1} + P_{t+1}) \right] & \text{for } R_{t+1} < R_t^f \text{ and } r_t > 1 \end{cases} \quad (8.8)$$

We can calculate the Sharpe Ratio as usual.

$$SR = \left| \frac{E[R] - R^f}{\sigma_R} \right|$$

8.3 Solution

Now that we have re-defined all necessary formulas we can begin to look for a solution to this model. We use the reference model of chapter 4 to generate

the consumption c_{t+1}^{\sim} , that is:

$$\max_{\tilde{c}_t} E\left[\sum_{t=0}^{\infty} \beta^t \ln \tilde{c}_t\right]$$

subject to the dynamics

$$\begin{aligned} k_{t+1} &= z_t A k_t^\alpha - c_t \\ \ln z_{t+1} &= \rho \ln z_t + \epsilon_t \end{aligned}$$

ϵ_t is still an i.i.d. random variable, $A = 5$, $\alpha = 0.34$, $\rho = 0.9$ and $\beta = 0.95$, see chapter 4, (4.1). The control function remains $\tilde{c}_t = (1 - \alpha\beta) A z_t k_t^\alpha$. The parameters for the loss aversion model are chosen to be

$$\lambda = 10, \eta = 0.9 \text{ and } k = 3.$$

Since the sets of datas for the stock return R_{t+1} and R_{t+1}^f are given, and those datas were generated with DP simulations, the standard deviation σ_ϵ was restricted to the interval $[-0.032, 0.032]$.

The MATLAB codes reading the parameters out of the file and based on those datas calculating the Sharpe Ratios and other characteristics are given in Appendix A.6.. The data sets are on the CD. We obtain $E[R]$, $E[R^f]$ and σ_R by the following equations:

$$E[R] = \frac{1}{N} \sum_{t=1}^N R_{t+1} \quad (8.9)$$

$$E[R] = \frac{1}{N} \sqrt{\sum_{t=1}^N (R_{t+1} - E[R])^2} \quad (8.10)$$

$$E[R^f] = \frac{1}{N} \sum_{t=1}^N R_{t+1}^f, \quad (8.11)$$

where N is the number of datas for R_{t+1} or R_{t+1}^f , i.e. $N = 50343$.

We will now, vary the parameter b_0 and then calculate the different Sharpe Ratios and other financial measures. The results are given in Table 7.1.. Please note that for $b_0 = 0$ we obtain the basic consumption model as given

| b_0 | 0 | 0.01 | 0.05 | 0.1 | 0.5 | 3 | 10 |
|------------|--------|--------|--------|--------|--------|--------|--------|
| $E[R]$ | 1.0527 | 1.0527 | 1.0537 | 1.0543 | 1.0563 | 1.0575 | 1.0577 |
| σ_R | 0.0089 | 0.0089 | 0.0091 | 0.0093 | 0.0100 | 0.0105 | 0.0106 |
| R^f | 1.0527 | 1.0527 | 1.0527 | 1.0527 | 1.0527 | 1.0527 | 1.0527 |
| SR | 0.0059 | 0.0033 | 0.1033 | 0.1735 | 0.3548 | 0.4499 | 0.4683 |

Table 8.1: Results for varying b_0

in our reference model. First thing we note is that even for $b_0 = 0$, which should be the usual consumption based model we used throughout chapter 5 to 7, we do not get the same results. Remember the Sharpe Ratio for the time series approach was 0.0091 and for the random variable approach it was 0.0080. The difference between the Sharpe Ratio we get in this context and the others must be due to the fact that another method for generating the returns was used. This can also be seen very easily in the expected value for the return of the risky asset.

We will now look at the outcomes for varying b_0 as can be found in Grüne and Semmler, [8], p.12. and represented in Table 8.2.. As we have already

| b_0 | 0 | 0.01 | 0.05 | 0.1 | 0.5 | 3 | 10 |
|----------|---------|---------|---------|---------|---------|---------|---------|
| $E[R]$ | 1.05273 | 1.05282 | 1.05362 | 1.05396 | 1.05419 | 1.05338 | 1.05297 |
| $Var[R]$ | 0.00840 | 0.00840 | 0.00848 | 0.00855 | 0.00858 | 0.00879 | 0.00842 |
| R^f | 1.05273 | 1.05273 | 1.05273 | 1.05273 | 1.05273 | 1.05273 | 1.05273 |
| SR | 0.00798 | 0.01114 | 0.10516 | 0.14453 | 0.17072 | 0.07438 | 0.02919 |

Table 8.2: Results for varying b_0 as in Güne and Semmler

seen in the past chapter, the results we get by time series approaches vary considerably from results obtained by the stochastic dynamic programming approach.

Chapter 9

Summary and Conclusion

The objective of this thesis was to compare three different methods for solving a stochastic growth model for asset price characteristics with each other. The results yielded by the time series and the random variable methods were generated in this thesis in chapter 5 and 6 respectively, but the ones of the stochastic dynamic programming approach were taken out of the paper by Grüne and Semmler, Solving Asset Pricing Models with Stochastic Dynamic Programming, [7]

By comparison we figured out that the values we get when applying the time series method to our model differ to the values we get when we use the random variable or the stochastic dynamic programming approach. In the latter two models we find equal results.

When analyzing the random variable and time series approach for differences we also compared the impact of the dynamics to our optimal control function. To express those relationships we calculated the correlations and covariances. As a result we found that the differences were rather large. Whereas a strong stochastic relationship between the capital and the consumption path could be found in the time series approach we do not find such a thing in the random variable approach. In fact, for the latter method we do not find any stochastic relationship between capital and consumption.

After looking for differences we studied the impact of the standard deviation of our shock variable on the model outcomes. As a result we figured out

that for small increases in its standard deviation the outcomes improve. But we also see that this behaviour does not hold for large standard deviations, because then the time series results start to underestimate the Sharpe Ratios.

Finally, we adjusted our stochastic growth model for loss aversion. In this context we just compared the results given by the time series approach and the stochastic dynamic programming approach and again figured out that those results differ.

Appendix A

Matlab Codes

The following MATLAB codes are based on the algorithms for generating pseudo random variables as listed in "Finanzderivate mit MATLAB" by Günther and Jünger. [10].

A.1 MATLAB codes for Path k

The following lagged Fibonacci-generator code gives us identically distributed random variables which is the first step to the identically and normally distributed random variables we need in the algorithms for the pictures of our reference model as well as for the solution of it.

```
function [dU]= Rwfibonacci_path(nu,mu,M,N,anzahl)
%-----
%Generator for Pseudo Random Variables, uniformly distributed on the
%interval [0,1]
%Rwfibonacci_path(nu,mu,M,N)
%nu~mu, M: Modulofactor, N: Time horizon
%-----
max=max(mu,nu);
X=M*rand(anzahl,max); %Matrix with random numbers
%-----
for i=1:anzahl
    for j=1+max:N+max
```



```

        X(i,j)=mod(X(i,j-mu)-X(i,j-nu),M);
        dU(i,j-max)=X(i,j)/M;
    end
end

```

The random variables of the lagged Fibonacci-generator work as an input to the Box-Muller algorithm used to derive standard normally distributed random variables with mean 0 and standard deviation 0.008.

```

function [Eps]= boxmuller_path(nu,mu,M,N)
%-----
%generates N(0,1)-distributed random variables
%boxmuller(nu,mu,M,N)
%nu =mu, M: Modulofactor, N: Time horizon
%-----
DU1=RWfibonacci_path(nu,mu,M,N,anzahl);
DU2=RWfibonacci_path(nu,mu,M,N,anzahl);
fori=1:anzahl
    Z(i,:)=sqrt(-2*log(DU1(i,:))).*cos(2*pi*DU2(i,:));
    Eps(i,:)=0.008.*Z(i,:); %garantees standard deviation 0.008
end

```

We need those standard normally distributed random variables for the k -paths of our reference model in chapter 5. The path to the stochastic process z , i.e. the labour process, is generated by the following MATLAB code.

```

function [z]= Refz_path(anzahl,N,y0)
%-----
%Paths for z based on our reference model
%Refz_path(anzahl,N,y0)
%N: Time horizon
%anzahl: Number of paths
%y0: Start value for y=log(z)
%-----
rho=0.9; %parameter out of the
          reference model
%-----

```

```

rho=0.9;
M=2048; nu=17; mu=5;
Epsilon=boxmuller_path(nu,mu,M,N,anzahl);           %matrix with random
                                                    variables
fori=1:anzahl
    y2(i,1)=y0;                                     %every path starts
                                                    with the staring value
    forj=1:N-1
        y2(i,j+1)=rho*y2(i,j)+Epsilon(i,j);
    end
end
z=exp(y2);

```

The plots for paths k in chapter 5 are generated by the following MATLAB code. z is used as an input variable.

```

function [c]= Refk_picture(anzahl,N,k01,k02,y01,y02,y03)
%-----
%Paths for k based on our reference model
%Refk(anzahl,N,k01,k02,y01,y02,y03)
%N: Time horizon
%anzahl: Number of paths
%k01,k02: start values for k
%y01,y02,y03: Start values for y=log(z)
%-----
A=5; alpha=0.34; rho=0.9; beta=0.95;
%-----
%Computation of k11 and k12 (start value y01)
for i=1:anzahl
    k11(i,1)=k01;
    z1=Refz_path(anzahl,N,y01);
    for j=1:N
        c11(i,j)=(1-alpha*beta)*A*z1(i,j)*k11(i,j)^alpha;
        k11(i,j+1)=z1(i,j)*A*k11(i,j)^alpha-c11(i,j);
    end
    k12(i,1)=k02;
    z2=Refz_path(anzahl,N,y01);
    for j=1:N
        c12(i,j)=(1-alpha*beta)*A*z2(i,j)*k12(i,j)^alpha;

```

```

        k12(i,j+1)=z2(i,j)*A*k12(i,j)^alpha-c12(i,j);
    end
end
%Building the ''average'' path
if anzahl==1
    k11=k11;
    k12=k12;
else
    k11=1/(anzahl+1)*sum(k11);
    k12=1/(anzahl+1)*sum(k12);
end;
%-----
%Computation of k21 and k22 (start value y02)
for i=1:anzahl
    k21(i,1)=k01;
    z1=Refz_path(anzahl,N,y02);
    for j=1:N
        c21(i,j)=(1-alpha*beta)*A*z1(i,j)*k21(i,j)^alpha;
        k21(i,j+1)=z1(i,j)*A*k21(i,j)^alpha-c21(i,j);
    end
    k22(i,1)=k02;
    z2=Refz_path(anzahl,N,y02);
    for j=1:N
        c22(i,j)=(1-alpha*beta)*A*z2(i,j)*k22(i,j)^alpha;
        k22(i,j+1)=z2(i,j)*A*k22(i,j)^alpha-c22(i,j);
    end
end
%Building the ''average'' path
if anzahl==1
    k21=k21;
    k22=k22;
else
    k21=1/(anzahl+1)*sum(k21);
    k22=1/(anzahl+1)*sum(k22);
end;
%-----
%Computation of k31 and k32 (start value y03)
for i=1:anzahl
    k31(i,1)=k01;
    z1=Refz_path(anzahl,N,y03);

```

```

for j=1:N
    c31(i,j)=(1-alpha*beta)*A*z1(i,j)*k31(i,j)^alpha;
    k31(i,j+1)=z1(i,j)*A*k31(i,j)^alpha-c31(i,j);
end
k32(i,1)=k02;
z2=Refz_path(anzahl,N,y03);
for j=1:N
    c32(i,j)=(1-alpha*beta)*A*z2(i,j)*k32(i,j)^alpha;
    k32(i,j+1)=z2(i,j)*A*k32(i,j)^alpha-c32(i,j);
end
end
%Building the 'average' path
if anzahl==1
    k31=k31;
    k32=k32;
else
    k31=1/(anzahl+1)*sum(k31);
    k32=1/(anzahl+1)*sum(k32);
end;
%-----
%Plots
subplot(311)
plot(0:N),k11,'r-',(0:N),k12,'k:')
subplot(312)
plot(0:N),k21,'r-',(0:N),k22,'k:')
subplot(313)
plot(0:N),k31,'r-',(0:N),k32,'k:')

```

A.2 MATLAB codes for the Time Series solution

For the Time Series solution of our basic stochastic growth model we will just need to evaluate one path for k and z due to convergence reasons. Therefore it is possible to simplify the above algorithms.

```
function [dU]= RWfibonacci(nu,mu,M,N)
```

```

%-----
%Generator for Pseudo Random Variables, uniformly distributed on the
%interval [0,1]
%RWfibonacci(nu,mu,M,N)
%nu =mu, M: Modulofactor, N: Time horizon
%-----
rand('seed',2)
max=max(mu,nu);
X=M*rand(1,max);           %(1 x max)-Matrix
                           with random variables
%-----
for j=1+max:N+max
    X(j)=mod(X(j-mu)-X(j-nu),M);
    dU(j-max)=X(j)/M ;
end

```

The random variables of the lagged Fibonacci-generator work as an input to the Box-Muller algorithm used to derive standardnormally distributed random variables with mean 0 and standard deviation 0.008.

```

function [Eeps]= boxmuller(nu,mu,M,N)
%-----
%generates N(0,1)-distributed random variables
%boxmuller(nu,mu,M,N)
%nu =mu, M: Modulofactor, N: Time horizon
%-----
DU1=RWfibonacci(nu,mu,M,N);
DU2=RWfibonacci(nu,mu,M,N);
Z=sqrt(-2*log(DU1).*cos(2*pi*DU2));
Eps=0.008.*Z;
Eeps=Eeps;

function [z]= Refz(N,y0)
%-----
%Paths for z based on our reference model
%Refz(N,y0)
%N: Time horizon

```

```

%y0: Start value for y=log(z)
%-----
rho=0.9; %parameter out of the
reference model

M=2048; nu=17; mu=5;
Epsilon=boxmuller(nu,mu,M,N); %matrix with random
variables
%-----
y2(1)=y0; %every paths starts at the
starting value

for j=1:N
    y2(j+1)=rho*y2(j)+Epsilon(j);
end
z=exp(y2);

function [c]= Refk(N,k0,y0)
%-----
%evaluates path k according to the refence model
%Refk(N,k0,y0)
%Rahel Berkemann
%
%N: time horizon
%k0: start value for path k
%y0: start value for path y
%-----
A=5; alpha=0.34; rho=0.9; beta=0.95;
%-----
%Calculating k by the use of the control function c
k(1)=k0;
z=Refz1(N,y0);
for j=1:N
    c(j)=(1-alpha*beta)*A*z(j)*k(j)^alpha;
    k(j+1)=z(j)*A*k(j)^alpha-c(j);
end
c(N+1)=(1-alpha*beta)*A*z(N+1)*k(N+1)^alpha;

```

We get the consumption path as the output variable, because the consumption path is what we need to calculate prices and returns and so the Sharpe

Ratio. The main MATLAB code we need for our analysis is the following, where we calculate the Upper Bound and the Sharpe Ratio.

```
function SR(N,k0,y0)
%-----
%Computation of the upper bound SB and the Sharpe Ratio for the
%Time Series approach
%SR(N,k0,y0)
%Raheel Berkemann
% %N: Time horizon
%k0: start value for k
%y0: Start value for y=log(z)
%-----
A=5; alpha=0.34; rho=0.9; beta=0.95;
%-----
%Computation of c
c=Refk(N,k0,y0);
%-----
%Computation of m(x)with x=(k,z)
for i=1:N-1
    m_t(i)=beta*c(i)/c(i+1);
end
%-----
% Computation of E(m(x))
Exp_m=1/(N-1)*sum(m_t);
%-----
%Computation of the risk-free interest rate R_f(x)
R_f=1./Exp_m;
%-----
%Computation of the Variance of m(x)
Var_m_t=(m_t-Exp_m*ones(1,N-1)).^2;
Var_m=1/(N-1)*sum(Var_m_t)
%-----
%Computation of the Standard deviation of m(x)
SD_m=sqrt(Var_m);
%-----
%Computation of the Upper Bound SB
SB=SD_m/Exp_m;
%-----
%Computing Sharpe Ratios
```

```

%Computation of p(x) with d(x)=c(x)
p=beta/(1-beta)*c;
%-----
%Computation of R(x)
for i=1:N-1
    R_t(i)=(c(i+1)+p(i+1))./p(i);
end
%-----
%Computing E(R(x))
Exp_R=1/(N-1)*sum(R_t);
%-----
%Computation of the Covariance of R(x) and m(x);
Cov_R_m_t=(R_t-Exp_R*ones(1,N-1)).*m_t-Exp_m*ones(1,N-1));
Cov_R_m=1/(N-1)*sum(Cov_R_m_t);
%Computation of Var(R(x)) and SD(R(x))
Var_R_t=(R_t-Exp_R*ones(1,N-1)).^2;
Var_R=1/(N-1)*sum (Var_R_t);
SD_R=sqrt(Var_R);
%-----
%Computation of the Sharpe Ratio via the Covariance formula
SR_Cov=(-R_f*Cov_R_m)/SD_R
%-----
%Computation of the Sharpe Ratio via the standard formula
SR_stand=(Exp_R-R_f)/SD_R

```

A.3 MATLAB codes for the Random Variable solution

The second approach for solving the reference model was to look at the variable $m(x)$ as being a random variable itself. The MATLAB codes to that approach are given in this section. I start with the main code for the Sharpe Ratio analysis. All additional functions, i.e. the codes for the computation of the expectations, variances and covariances, are listed afterwards. Since we need this programme in chapter 7 as well when we look for different outcomes depending on σ_ϵ we include a variable s for the standard deviation and a for the interval boundaries.

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I start with the programme for figuring out the interval boundaries, which just worked by looking at the plot computed by the following programme.

```
function density(s,a)
%-----
%plots the density function for epsilon
%density(s,a)
%s: standard deviation for epsilon
%a: interval boundary
%-----
m=a/20;
epsilon=(-a:m:a);
density=1/(sqrt(2*pi)*s).*exp(-1/2*(epsilon.^2./(s^2)));
plot(epsilon,density,'r')
ylabel('f(epsilon)')
xlabel('epsilon')
```

Now, the main programme for calculating the Sharpe Ratios in the random variable model is given. "quad" is a MATLAB internal routine that calculates the integral by using Simpson's Quadrature. The command `scale=quad(@Exp_m,-a,a,1.e-9,[],y0,k0,s)`; loads the programme `Exp_m`, it calculates the integral in the interval $[-a, a]$, `1.e-9` is the accuracy that `quad` should use for computation, `y0`, `k0` are the initial values and `s` is the parameter for the standard variable.

```
function RV(y0,k0,s,a)
%-----
%Computing random variable m
%RV(y0,k0)
%Raheel Berkemann
%
%y0: start value for y
%k0: start value for k
%-----
beta=0.95; alpha=0.34; rho=0.9; A=5;
%-----
scale=quad(@Exp_m,-a,a,1.e-9,[],y0,k0,s);
Exp_m=quad(@Exp_m,-a,a,1.e-9,[],y0,k0,s,scale)
Var_m=quad(@Var_m,-0.032,0.032,1.e-9,[],y0,k0,s,scale,Exp_m);
```

```

SD_m=sqrt(Var_m);
R_f=1/Exp_m
Exp_R=quad(@Exp_R,-a,a,1.e-9,[],y0,k0,s,scale)
Var_R=quad(@Var_R,-a,a,1.e-9,[],y0,k0,s,scale,Exp_R);
SD_R=sqrt(Var_R);
%-----
%Computing the Upper Bound
SB=SD_m/Exp_m
%-----
%Computing the Sharpe Ratio
%1. SR_Cov
Cov_R_m=quad(@Cov_R_m,-a,a,1.e-9,[],y0,k0,s,scale,Exp_m,Exp_R);
SR_Cov=(-R_f*Cov_R_m)/SD_R
%2. SR_stand
SR_stand=abs((Exp_R-R_f/SD_R)

```

The following programmes that are listed are the ones evoked by the main programm.

```

function [m]= Exp_m(x,y0,k0,s,scale)
%-----
beta=0.95; alpha=0.34; rho=0.9; A=5;
%-----
c0=(1-alpha*beta)*A*exp(y0)*k0^alpha;
y1=rho*y0+x;
k1=exp(y0)*A*k0^alpha-c0;
c1=(1-alpha*beta)*A*exp(y1)*k1^alpha;
m=beta*c0./c1.*(1/sqrt(2*pi)*s).*(1/scale)
  .*exp(-(x.^2./(2*s^2)));

```

```

function [m]= Var_m(x,y0,k0,s,scale,Exp_m)
%-----
beta=0.95; alpha=0.34; rho=0.9; A=5;
%-----
c0=(1-alpha*beta)*A*exp(y0)*k0^alpha;
y1=rho*y0+x;
k1=exp(y0)*A*k0^alpha-c0;
c1=(1-alpha*beta)*A*exp(y1)*k1^alpha;

```

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```
m=(beta*c0./c1-Exp_m).^2.*(1/scale)
.*(1/sqrt(2*pi)*s).*exp(-(x.^2./(2*s^2))));
```

```
function [R]= Exp_R(x,y0,k0)
%-----
beta=0.95; alpha=0.34; rho=0.9; A=5;
%-----
c0=(1-alpha*beta)*A*exp(y0)*k0^alpha;
y1=rho*y0+x;
k1=exp(y0)*A*k0^alpha-c0;
c1=(1-alpha*beta)*A*exp(y1)*k1^alpha;
p0=beta/(1-beta)*c0;
p1=beta/(1-beta)*c1;
R=(c1+p1)./p0.*(1/sqrt(2*pi)*0.008).*
(1/scale).*exp(-(x.^2./(2*0.008^2))));
```

```
function [R]= Var_R(x,y0,k0,Exp_R)
%-----
beta=0.95; alpha=0.34; rho=0.9; A=5;
%-----
c0=(1-alpha*beta)*A*exp(y0)*k0^alpha;
y1=rho*y0+x;
k1=exp(y0)*A*k0^alpha-c0;
c1=(1-alpha*beta)*A*exp(y1)*k1^alpha;
p0=beta/(1-beta)*c0;
p1=beta/(1-beta)*c1;
R=((c1+p1)./p0-Exp_R).^2.*(1/scale)
.*(1/sqrt(2*pi)*0.008).*exp(-(x.^2./(2*0.008^2))));
```

```
function [R]= Cov_R.m(x,y0,k0,Exp_m,Exp_R)
%-----
beta=0.95; alpha=0.34; rho=0.9; A=5;
%-----
c0=(1-alpha*beta)*A*exp(y0)*k0^alpha;
y1=rho*y0+x;
k1=exp(y0)*A*k0^alpha-c0;
c1=(1-alpha*beta)*A*exp(y1)*k1^alpha;
```

```

p0=beta/(1-beta)*c0;
p1=beta/(1-beta)*c1;
R=((c1+p1)./p0-Exp_R).*(beta*c0./c1-Exp_m).*(1/scale)
  .*(1/sqrt(2*pi)*0.008).*exp(-(x.^2./(2*0.008^2)));

```

A.4 MATLAB codes for the Check

The programme to check for correctness as discussed in chapter 7.1 looks similar to the programme used in chapter 5. The only things we change were already discussed in chapter 7.1.

```

function SR_control(N,y0,k0)
%-----
%checks for correctness of the code SR(N,y0,k0)
%Time Series approach
%SR(N,k0,y0)
%Rahel Berkemann
% %N: Time horizon
%k0: start value for k
%y0: Start value for y=log(z)
%-----

%Lagged Fibonacci
M=2048; nu=17; mu=5;
rand('seed',2);
X1=M*rand(1,max(mu,nu));
X2=M*rand(1,max(mu,nu));
for i=max(mu,nu)+1:N+max(mu,nu)
    X1(i)=mod(X1(i-mu)-X1(i-nu),M);
    U1(i-max(mu,nu))=X1(i)/M;
    X2(i)=mod(X2(i-mu)-X2(i-nu),M);
    U2(i-max(mu,nu))=X2(i)/M;
end
%-----

%boxmuller Z=sqrt(-2*log(U1)).*cos(2*pi*U2);
E_Z=1/N*sum(Z);

```

```

Var_Z=1/N*sum((Z-E_Z*ones(1,N)).^2);
SD_Z=sqrt(Var_Z);
%-----

%RV with mean 0 and SD 0.008
Eps=0.008*Z;
E_Eps=1/N*sum(Eps);
Var_Eps=1/N*sum((Eps-E_Eps*ones(1,N)).^2);
SD_Eps=sqrt(Var_Eps);
%-----

%Initialising parameters for the stochastic growth model
beta=0.95; alpha=0.34; rho=0.9; A=5;
%-----

%Calculating path z
y(1)=y0;
for t=1:N
    y(t+1)=rho*y0+Eps(t);
end
z=exp(y);
%-----

%Computing k and c
k(1)=k0;
c(1)=(1-alpha*beta)*z(1)*A*k(1)^alpha;
k(2)=exp(y0)*A*k0^alpha-c(1);
for t=2:N
    c(t)=(1-alpha*beta)*z(t)*A*k(2)^alpha;
end
%-----

%Computing m
for t=1:N
    m_t(t)=beta*(c(1)/c(t+1));
end
Exp_m=1/N*sum(m_t)
Var_m=1/N*sum((m_t-Exp_m*ones(1,N)).^2);
SD_m=sqrt(Var_m)
%-----

```

```

%Computing the Upper Bound S_B
S_B=SD_m/Exp_m
%-----

%Risk-free interest rate
R_f=1/Exp_m
%-----

%Computing p(x) with d(x)=c(x)
p=beta/(1-beta)*c;
%-----

%Computing R(x)
for i=1:N
    R_t(i)=(c(i+1)+p(i+1))/p(1);
end
%-----

%Computing E(R(x))
Exp_R=1/N*sum(R_t);
%-----

%Computing Covariances
Cov_R_m_t=(R_t-Exp_R*ones(1,N)).*(m_t-Exp_m*ones(1,N));
Cov_R_m=1/N*sum(Cov_R_m_t);
%-----

%Computing Var(R(x)) and SD(R(x))
Var_R_t=(R_t-Exp_R*ones(1,N)).^2;
Var_R=1/N*sum(Var_R_t);
SD_R=sqrt(Var_R);
%-----

%Computing Sharpe Ratio via Covariance
SR_cov=(-R_f*Cov_R_m)/SD_R
%-----

%Computing Sharpe Ratio with standard formula
SR_stand=abs((Exp_R-R_f)/SD_R)

```

A.5 MATLAB codes for Correlations

We have one main programm giving us the Correlations and Covariances for our analysis in chapter 7. The first part of the following programm gives us the Correlations and Covariances for the time series case, whereas the second part gives us the ones for the random variable approach. For the latter additional programmes are used for computing the integrals needed for the calculation of covariances in the random variable case. The input paramter s for the standard deviation needed in the random variable part is set to 0.008. When we calculate correlations for another standard deviation the boxmuller.m programme used in the time series part needs to be adjusted, i.e. $0.008*Z$ has to be changed to $s*Z$.

```
function cor(N,y0,k0,s,a)
%-----
%Calculates correlations between time series %cor(N,k0,y0,s,a)
%Raehl Berkemann
% %N: Time horizon
%k0: start value for k
%y0: Start value for y=log(z)
%-----

%Initialising parameters for the stochastic growth model
beta=0.95; alpha=0.34; rho=0.9; A=5;
%-----

%Expectation and Variance of Epsilon
M=2048; nu=17; mu=5;
Eps=boxmuller(nu,mu,M,N);
E_Eps=1/N*sum(Eps);
V_Eps=1/N*sum((Eps-E_Eps).^2);
%-----

%Covariance of z and c
z=Refz(N,y0);
E_z=1/(N+1)*sum(z);
Var_z_t=(z-E_z*ones(1,N+1)).^2;
Var_z=1/(N+1)*sum((z-E_z*ones(1,N+1)).^2);

c=Refk(N,k0,y0);
```

```

E_c=1/(N+1)*sum(c);
Var_c=1/(N+1)*sum((c-E_c*ones(1,N+1)).^2);

Cov_z_c_t=(z-E_z*ones(1,N+1)).*(c-E_c*ones(1,N+1));
Cov_z_c=1/(N+1)*sum(Cov_z_c_t);

Cor_z_c=Cov_z_c/(sqrt(Var_z)*sqrt(Var_c))

%-----

%Covariance of k and z
k2(1)=k0;
fort=1:N
    k2(t+1)=A*z(t)*k2(t)^alpha-c(t);
end
E_k=1/(N+1)*sum(k2);
Var_k=1/(N+1)*sum((k2-E_k*ones(1,N+1)).^2);

Cov_z_k_t=(z-E_z*ones(1,N+1)).*(k2-E_k*ones(1,N+1));
Cov_z_k=1/(N+1)*sum(Cov_z_k_t);

Cor_z_k=Cov_z_k/(sqrt(Var_z)*sqrt(Var_k))

%-----

%Covariance of p and z
%Berechnung von p(x) mit d(x)=c(x)
p=beta/(1-beta)*c;
E_p=1/(N+1)*sum(p);
Var_p=1/(N+1)*sum((p-E_p*ones(1,N+1)).^2);
Cov_z_p_t=(z-E_z*ones(1,N+1)).*(p-E_p*ones(1,N+1));
Cov_z_p=1/(N+1)*sum(Cov_z_p_t);

Cor_z_p=Cov_z_p/(sqrt(Var_z)*sqrt(Var_p))

%-----

%Covariance of k^alpha and c
k_alpha=k2.^alpha;
E_k_alpha=1/(N+1)*sum(k_alpha);
Var_k_alpha=1/(N+1)*sum((k_alpha-E_k_alpha*ones(1,N+1)).^2);

Cov_k_alpha_c_t=(c-E_c*ones(1,N+1)).*(k_alpha-E_k_alpha*ones(1,N+1));

```



```

Cov_k_alpha_c=1/(N+1)*sum(Cov_k_alpha_c_t);

Cor_k_alpha_c=Cov_k_alpha_c/(sqrt(Var_c)*sqrt(Var_k_alpha))
%-----
%-----

%Covariance in the random variable case
scale = quad(@scale,-a,a,1.e-15, [],s);

%Exp_Eps= quad(@Exp_Eps,-0.032,0.032,1.e-9, [],y0,k0)
%Covariance of z and c
Exp_z= quad(@Exp_z,-a,a,1.e-9, [],y0,k0,s,scale);
Var_z= quad(@Var_z,-a,a,1.e-9, [],y0,k0,s,scale,Exp_z);
Exp_c= quad(@Exp_c,-a,a,1.e-9, [],y0,k0,s,scale);
Var_c= quad(@Var_c,-a,a,1.e-9, [],y0,k0,s,scale,Exp_c);
Cov_z_c = quad(@Cov_z_c,-a,a,1.e-9, [],y0,k0,s,scale,Exp_z,Exp_c);

Cor2_z_c=Cov_z_c/(sqrt(Var_z)*sqrt(Var_c))
%-----

%Covariance of k and z
c0=(1-alpha*beta)*A*exp(y0)*k0^alpha;
k1=exp(y0)*A*k0^alpha-c0;
Exp_k=k1;
Var_k=(k1-Exp_k);
Cov_z_k = quad(@Cov_z_k,-a,a,1.e-9, [],y0,k0,s,scale,Exp_z,Exp_k)
%-----

%Covariance of p and z
Exp_p=beta/(1-beta)*Exp_c;
Var_p = (beta/(1-beta))^2*Var_c;
Cov_z_p= quad(@Cov_z_p,-a,a,1.e-9, [],y0,k0,s,scale,Exp_z,Exp_p);

Cor2_z_p=Cov_z_p/(sqrt(Var_z)*sqrt(Var_p))
%-----

%Covariance of k^alpha and c
c0=(1-alpha*beta)*A*exp(y0)*k0^alpha;
k1=exp(y0)*A*k0^alpha-c0;
Exp_k_alpha=k1^alpha;
Var_k_alpha=k1^alpha-Exp_k.alpha;

```

```
Cov_k_alpha_c =quad(@Cov_k_alpha_c,-a,a,1.e-9,[],y0,k0,s,scale,
                    Exp_c,Exp_k_alpha)
```

```
%-----
```

Next, the additional programmes needed in the random variable case, are listed. These programmes are loaded by the same commands that were explained in appendix A.3.

```
function [y]= Exp_z(x,y0,k0,s,scale)
%-----
%Initialisierungen
beta=0.95; alpha=0.34; rho=0.9; A=5;
%-----
c0=(1-alpha*beta)*A*exp(y0)*k0^alpha;
y1=rho*y0+x;
k1=exp(y0)*A*k0^alpha-c0;
c1=(1-alpha*beta)*A*exp(y1)*k1^alpha;
y=exp(y1).*(1/scale).*(1/(sqrt(2*pi)*s).*exp(-(x.^2./(2*s^2))));
```

```
function [y]= Var_y(x,y0,k0,s,scale,Exp_z)
%-----
%Initialisierungen
beta=0.95; alpha=0.34; rho=0.9; A=5;
%-----
c0=(1-alpha*beta)*A*exp(y0)*k0^alpha;
y1=rho*y0+x;
k1=exp(y0)*A*k0^alpha-c0;
c1=(1-alpha*beta)*A*exp(y1)*k1^alpha;
y=(exp(y1)-Exp_z).^2
.*(1/scale).*(1/(sqrt(2*pi)*s).*exp(-(x.^2./(2*s^2))));
```

```
function [c]= Exp_c(x,y0,k0,s,scale)
%-----
%Initialisierungen
beta=0.95; alpha=0.34; rho=0.9; A=5;
%-----
```

```

c0=(1-alpha*beta)*A*exp(y0)*k0^alpha;
y1=rho*y0+x;
k1=exp(y0)*A*k0^alpha-c0;
c1=(1-alpha*beta)*A*exp(y1)*k1^alpha;
c=c1.*(1/scale).*(1/(sqrt(2*pi)*s).*exp(-(x.^2./(2*s^2))));

```

```

function [c]= Var_c(x,y0,k0,s,scale,Exp_c)
%-----
%Initialisierungen
beta=0.95; alpha=0.34; rho=0.9; A=5;
%-----
c0=(1-alpha*beta)*A*exp(y0)*k0^alpha;
y1=rho*y0+x;
k1=exp(y0)*A*k0^alpha-c0;
c1=(1-alpha*beta)*A*exp(y1)*k1^alpha;
c=(c1-Exp_c).^2
  .*(1/scale).*(1/(sqrt(2*pi)*s).*exp(-(x.^2./(2*s^2))));

```

```

function [y]= Cov_z_c(x,y0,k0,s,scale,Exp_z,Exp_c)
%-----
%Initialisierungen
beta=0.95; alpha=0.34; rho=0.9; A=5;
%-----
c0=(1-alpha*beta)*A*exp(y0)*k0^alpha;
y1=rho*y0+x;
k1=exp(y0)*A*k0^alpha-c0;
c1=(1-alpha*beta)*A*exp(y1)*k1^alpha;
y=(exp(y1)-Exp_z).*(c1-Exp_c)
  .*(1/scale).*(1/(sqrt(2*pi)*s).*exp(-(x.^2./(2*s^2))));

```

```

function [y]= Cov_z_k(x,y0,k0,s,scale,Exp_z,Exp_k)
%-----
%Initialisierungen
beta=0.95; alpha=0.34; rho=0.9; A=5;
%-----
c0=(1-alpha*beta)*A*exp(y0)*k0^alpha;
y1=rho*y0+x;

```

```

k1=exp(y0)*A*k0^alpha-c0;
c1=(1-alpha*beta)*A*exp(y1)*k1^alpha;
y=(exp(y1)-Exp_z).*(k1-Exp_k)
  *(1/scale).*(1/(sqrt(2*pi)*s).*exp(-(x.^2./(2*s^2))));

```

```

function [y]= Cov_z_p(x,y0,k0,s,scale,Exp_z,Exp_p)
%-----
%Initialisierungen
beta=0.95; alpha=0.34; rho=0.9; A=5;
%-----
c0=(1-alpha*beta)*A*exp(y0)*k0^alpha;
y1=rho*y0+x;
k1=exp(y0)*A*k0^alpha-c0;
c1=(1-alpha*beta)*A*exp(y1)*k1^alpha;
y=(exp(y1)-Exp_z).*(beta/(1-beta)*c1-Exp_p)
  *(1/scale).*(1/(sqrt(2*pi)*s).*exp(-(x.^2./(2*s^2))));

```

```

function [y]= Cov_k_alpha_c(x,y0,k0,s,scale,Exp_c,Exp_k_alpha)
%-----
%Initialisierungen
beta=0.95; alpha=0.34; rho=0.9; A=5;
%-----
c0=(1-alpha*beta)*A*exp(y0)*k0^alpha;
y1=rho*y0+x;
k1=exp(y0)*A*k0^alpha-c0;
c1=(1-alpha*beta)*A*exp(y1)*k1^alpha;
y=(c1-Exp_c).*(k1.^alpha-Exp_k_alpha)
  *(1/scale).*(1/(sqrt(2*pi)*s).*exp(-(x.^2./(2*s^2))));

```

A.6 MATLAB code for the Loss Aversion model

The following MATLAB codes calculate the return of risky assets and the risk-free return by loading a file where the given return datas are stored. This is done by the command 'load'. Since the only thing that is varied is the factor b_0 , just one "example" programme is given in this chapter. The

parameter j , $j \in \{0, 001, 005, 01, 05, 3, 10\}$, therefore denotes the factor used for b_0 , where for example 001 means 0.01. In order to load the files as stated in the programme, they have to be stored in the same folder as the programme. All programmes and files are on the included CD.

```
function bj
T = load('bj_Reihe.asc');
R_t=T(:,8);
R_f_t=T(:,9);
N=length(R_t);
%-----
%Computing E(R(x));
Exp_R=1/N*sum(R_t)
Var_R_t=(R_t-Exp_R*ones(N,1)).^2;
Var_R=1/N*sum(Var_R_t);
SD_R=sqrt(Var_R)
%-----
%Computing R_f
R_f=1/N*sum(R_f_t)
%-----
%Computing Sharpe Ratio with standard formula
SR_stand=abs((Exp_R-R_f)/SD_R)
```

Appendix B

The included CD

The CD included stores all relevant programmes and files used in this thesis. The set-up as follows: main folder is called Methods for Calculating Sharpe Ratios and contains the following folders and programmes:

1. Folder Path k
 - (a) RWfibonacci_path.m
 - (b) boxmuller_path.m
 - (c) Refz_path.m
 - (d) Refk_picture.m
2. Folder Time Series approach
 - (a) RWfibonacci.m
 - (b) boxmuller.m
 - (c) Refz.m
 - (d) Refk.m
 - (e) SR.m
3. Folder Random Variable approach
 - (a) RV.m

- (b) density.m
- (c) scale.m
- (d) Exp_m.m
- (e) Var_m.m
- (f) Exp_R.m
- (g) Var_R.m
- (h) Cov_R.m.m

4. Folder Check

- (a) SR_control.m

5. Folder Correlations

- (a) cor.m
- (b) RWfibonacci.m
- (c) boxmuller.m
- (d) Refz.m
- (e) Refk.m
- (f) scale.m
- (g) Exp_z.m
- (h) Var_z.m
- (i) Exp_c.m
- (j) Var_c.m
- (k) Cov_z_c.m
- (l) Cov_z_p.m
- (m) Cov_z_k.m
- (n) Cov_k_alpha_c.m

6. Folder Loss Aversion model

- (a) data files b0_reihe, b001_reihe, b005_reihe, b01_reihe, b05_reihe, b3_reihe and b10_reihe.
- (b) programmes b0.m, b001.m, b005.m, b01.m, b05.m, b3.m and b10.m.

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Erklärung

Hiermit erkläre ich, dass die vorliegende Arbeit selbstständig und unter Verwendung der angegebenen Quellen und Hilfsmittel angefertigt habe.

Diese Arbeit hat in gleicher oder ähnlicher Form noch keiner anderen Prüfungsbehörde vorgelegen.

Bayreuth, den 13. Juni 2006

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Rahel Berkemann